

Boundary WZW, G/H , G/G and CS theories

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Abstract

We extend the analysis [18] of the canonical structure of the Wess-Zumino-Witten theory to the bulk and boundary coset G/H models. The phase spaces of the coset theories in the closed and in the open geometry appear to coincide with those of a double Chern-Simons theory on two different 3-manifolds. In particular, we obtain an explicit description of the canonical structure of the boundary G/G coset theory. The latter may be easily quantized leading to an example of a two-dimensional topological boundary field theory.

1 Introduction

Bidimensional boundary conformal field theory is a subject under intense study in view of its applications to boundary phenomena in 1+1- or two-dimensional critical systems and to the brane physics in string theory. Although much progress has been achieved in understanding boundary CFT's since the seminal paper of Cardy [6], much more remains to be done. The structure involved in the boundary theories is richer than in the bulk ones and their classification program involves new notions and an interphase with sophisticated mathematics [27][29]. One approach that offered a conceptual insight into the properties of correlation functions of boundary conformal models consisted of relating them to boundary states in three-dimensional topological field theories [11][12]. In the simplest case of the boundary Wess-Zumino-Witten (WZW) models (conformal sigma models with a group G as a target [33]), the topological three-dimensional model appears to be the group G Chern-Simons (CS) gauge theory [31][34]. In [18] it has been shown how the relation between the boundary WZW model and the CS theory arises in the canonical approach.

The purpose of the present paper is to extend the analysis of [18] to the case of the coset G/H models of conformal field theory obtained by gauging in the group G WZW model the

adjoint action of a subgroup $H \subset G$. In the WZW model the simplest class of boundary conditions is obtained by restricting the boundary values of the classical G -valued field g to fixed conjugacy classes in the group labeled by weights of the Lie algebra \mathfrak{g} of G . Such boundary conditions reduce to the Dirichlet conditions for toroidal targets. It was shown in [18] that the phase space of the WZW model on a strip with such boundary conditions is isomorphic to the phase space of the CS theory on the time-line \mathbf{R} times a disc D with two time-like Wilson lines. In the coset models we shall use more general boundary conditions requiring that field g belongs on the boundary components to pointwise product of group G and subgroup H conjugacy classes. The phase space of the coset theory on a strip with such boundary conditions becomes isomorphic to the phase space of the double CS theory on $\mathbf{R} \times D$ with group G and group H gauge fields, both coupled to two time-like Wilson lines. In particular, the phase space of the boundary G/G coset model¹ becomes isomorphic to the moduli space of flat connections on the 2-sphere with four punctures. The latter case lends itself easily to quantization giving rise to an example of the two-dimensional boundary topological field theory, a structure that promises to play the role of the K -theory of loop spaces [27].

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2 Action functionals of the WZW and coset theories

The Wess-Zumino-Witten model is a specific two-dimensional sigma model with a group manifold G as the target. For simplicity, we shall assume that G is compact connected and simply connected. We shall denote by \mathfrak{g} the Lie algebra of G . The G -valued fields of the WZW model are defined on two-dimensional surfaces Σ (the “worldsheets”) that we shall take oriented and equipped with a conformal or pseudo-conformal structure. The action of the model in the Euclidean signature is the sum of two terms:

$$S(g) = \frac{k}{4\pi i} \int_{\Sigma} \text{tr}(g^{-1} \partial g)(g^{-1} \bar{\partial} g) + S^{WZ}(g). \quad (2.1)$$

Above, tr stands for the properly normalized Killing form. The second (Wess-Zumino) term in the action is related to the canonical closed 3-form $\chi(g) = \frac{1}{3} \text{tr}(g^{-1} dg)^3$ on G . Informally, it may be written as

$$S^{WZ}(g) = \frac{k}{4\pi i} \int_{\Sigma} g^* \omega \quad (2.2)$$

where ω is a 2-form on G such that $d\omega = \chi$. This definition is, however, problematic since there is no global ω with the last property. If Σ has no boundary then the problem may

¹There are other ways to impose boundary conditions in the G/G theory [10] that permit to relate it to the boundary topological Poisson sigma models of [7]

be solved by setting [33]

$$S^{WZ}(g) = \frac{k}{4\pi i} \int_B \tilde{g}^* \chi, \quad (2.3)$$

where B is a 3-manifold such that $\partial B = \Sigma$ and \tilde{g} extends g to B . It is well known that this determines $S^{WZ}(g)$ modulo $2\pi i k$ so that the amplitudes $\exp[-S^{WZ}(g)]$ are well defined if k is integer. By the Stokes Theorem, the definition (2.3) reduces to the naive expression (2.2) whenever g maps into the domain of a local form ω . The variation $\delta S^{WZ}(g)$ involves only the 3-form χ so that the classical equations are determined unambiguously.

For surfaces with boundary, one should impose proper boundary conditions on fields g . Let $\partial\Sigma = \sqcup S_n^1$ where S_n^1 are disjoint circles always considered with the orientation inherited from Σ . Let, for μ in the Cartan subalgebra of \mathfrak{g} , C_μ denote the conjugacy class $\{\gamma e^{2\pi i \mu} \gamma^{-1} \mid \gamma \in G\}$. We shall require that

$$g(S_n^1) \subset C_{\mu_n}. \quad (2.4)$$

These are the so called fully symmetric conformal boundary conditions. When restricted to a conjugacy class C_μ , the 3-form χ becomes exact. In particular, the 2-form

$$\omega_\mu(g) = \text{tr}(\gamma^{-1} d\gamma) e^{2\pi i \mu} (\gamma^{-1} d\gamma) e^{-2\pi i \mu} \quad (2.5)$$

on C_μ satisfies $d\omega_\mu = \chi|_{C_\mu}$. Let $\Sigma' = \Sigma \# (\sqcup D_n)$ be the surface without boundary obtained from Σ by gluing discs D_n to the boundary components S_n^1 of Σ , see Fig. 1.

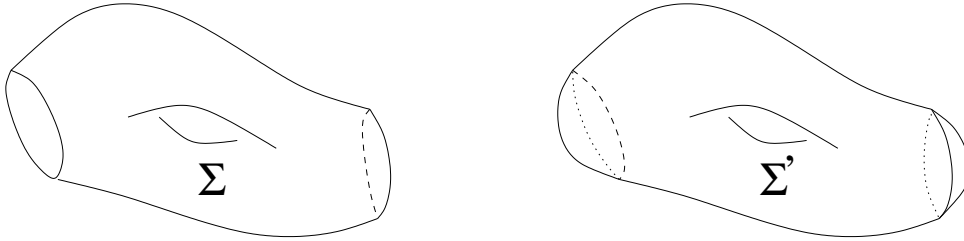


Fig. 1

Each field g satisfying the boundary conditions (2.4) may be extended to $g' : \Sigma' \rightarrow G$ in such a way that $g'(D_n) \subset C_{\mu_n}$ (the conjugacy classes are simply connected). Following [3][15], we shall define the WZ-action of the field g satisfying the boundary conditions (2.4) by setting

$$S^{WZ}(g) = S^{WZ}(g') - \frac{k}{4\pi i} \sum_n \int_{D_n} g'^* \omega_{\mu_n}. \quad (2.6)$$

This again reduces to the naive definition whenever g maps into the domain of a form ω such that $d\omega = \chi$, provided that the restrictions of ω to C_{μ_n} coincide with ω_{μ_n} . A different choice of the restrictions would change the boundary contributions to the classical equations. As explained in [15], (for $k \neq 0$) the right hand side of (2.6) is well defined modulo $2\pi i$ iff k is an integer and $C_{\mu_n} = C_{\lambda_n/k}$ for integrable weights² λ_n . The boundary conditions are thus labeled by the same set as the bulk primary fields of the current algebra also corresponding to integrable weights. This is an illustration of the Cardy's theory of boundary conditions [6].

The G/H coset theories [20] may be realized as the versions of the group G WZW theory where the adjoint action of the subgroup $H \subset G$ has been gauged [5][16][22][17]. Let A denote a 1-form with values in $i\mathfrak{h}$, where \mathfrak{h} stands for the Lie algebra of H . For $\partial\Sigma = \emptyset$, the action of the theory coupled to the gauge field $A = A^{10} + A^{01}$ is

$$S(g, A) = S(g) - \frac{k}{2\pi i} \int_{\Sigma} \text{tr} \left[(g\partial g^{-1})A^{01} + A^{10}(g^{-1}\bar{\partial}g) + gA^{10}g^{-1}A^{01} - A^{10}A^{01} \right]. \quad (2.7)$$

In fact, getting rid of the so called “fixed point problem” [30][13] (that obstructs factorization properties of the theory) requires considering the WZW theory coupled to gauge fields in non-trivial H/Z -bundles, where Z is the intersection of the center of G with H [28][21]. For simplicity, we shall not do it here. For the surfaces with boundary $\partial\Sigma = \sqcup S_n^1$, we shall use the same formula (2.7) to include the coupling to the gauge field, but we shall admit more general boundary conditions for the field g than the ones considered before. Namely, we shall assume that

$$g|_{S_n^1} = g_n h_n^{-1} \quad \text{with} \quad g_n : S_n^1 \rightarrow C_{\mu_n}^G, \quad h_n : S_n^1 \rightarrow C_{\mu_n}^H. \quad (2.8)$$

In other words, we shall admit fields g that, on each boundary component, are a pointwise product of loops in conjugacy classes of, respectively, group G and group H , keeping also track of the decomposition factors³. We shall label such conditions by pairs $(\mu_n, \nu_n) \equiv M_n$. For $\nu_n = 0$, they reduce to the conditions considered in the previous section. We still need to generalize the definition of the Wess-Zumino term of the action to fields g satisfying (2.8). Such fields may be extended to maps $g' : \Sigma' \rightarrow G$ in such a way that

$$g'|_{D_n} = g'_n h'_n{}^{-1} \quad \text{with} \quad g'_n|_{S_n^1} = g_n, \quad h'_n|_{S_n^1} = h_n$$

$$\text{and} \quad g'_n(D_n) \subset C_{\mu_n}^G, \quad h'_n(D_n) \subset C_{\nu_n}^H.$$

We shall define then

$$S^{WZ}(g) = S^{WZ}(g') - \frac{k}{4\pi i} \sum_n \int_{D_n} \left[g_n^* \omega_{\mu_n}^G - h_n^* \omega_{\nu_n}^H + \text{tr}(g_n'^{-1} dg'_n)(h_n'^{-1} dh'_n) \right]. \quad (2.9)$$

The form in the brackets has $(g'_n h_n'^{-1})^* \chi$ as the exterior derivative which assures invariance of the right hand side under continuous deformations of g'_n and h'_n inside discs D_n . If $k \neq 0$ and H is simply connected then a slight extension of the argument in [15] shows

²The weights integrable at level k are the ones lying in the positive Weyl alcove inflated by k

³The decomposition of elements of the pointwise product $C_{\mu_n}^G (C_{\nu_n}^H)^{-1}$ might not be unique.

that $S^{WZ}(g)$ given by (2.8) is well defined modulo $2\pi i$ iff k is integer and $C_{\mu_n}^G = C_{\lambda_n/k}^G$, $C_{\nu_n}^H = C_{\eta_n/k}^H$, where λ_n and η_n are integrable weights of \mathfrak{g} and \mathfrak{h} , respectively. We shall use (2.1), (2.9) and (2.7) to define the complete gauged action $S(g, A)$. With the above choices of the boundary labels, the gauge invariance

$$\exp[-S(hgh^{-1}, hAh^{-1} + hdh^{-1})] = \exp[-S(g, A)] \quad (2.10)$$

holds for $h : \Sigma \rightarrow H$. If H has an abelian factor, then the same selection of boundary conditions is imposed if we add to the demand that $\exp[-S^{WZ}(g)]$ be well defined the requirement of the gauge-invariance (2.10). For example, for the parafermionic $SU(2)/U(1)$ coset theory, this restricts the boundary labels to pairs $(\lambda_n, \eta_n) = (j_n\sigma_3, m_n\sigma_3)$ with $j_n = 0, \frac{1}{2}, \dots, \frac{k}{2}$ and $m_n = 0, \frac{1}{2}, \dots, k - \frac{1}{2}$. The labels of the parafermionic primary states (j, m) have additional selection rule $j = m \bmod 1$ and the identification $(j, m) \sim (\frac{k}{2} - j, m + \frac{k}{2} \bmod k)$. For the boundary labels, the first may be imposed by requiring the gauge invariance with respect to $h : \Sigma \rightarrow U(1)/\mathbf{Z}_2$ and the second by identifying the decompositions $g_nh_n^{-1}$ and $(-g_n)(-h_n)^{-1}$. Similarly, in the general case we may impose the local H/Z gauge invariance and identify the decompositions differing by an element in Z [19]. Such restrictions lead to the same labeling of the boundary conditions and of the primary fields, but is not obligatory if we ignore the fixed point problem.

Since the gauge field A enters quadratically into the action (2.7), it may be eliminated classically (and also quantum mechanically) from the equations of motion. What results is a sigma model with the space $G/Ad(H)$ of the orbits of the adjoint action of H on G as the target. The target space $G/Ad(H)$ (that may be singular) comes equipped with a specific metric, a non-metric volume form (“dilaton field”) and a 2-form. Let $[g]$ denote the projection of g to $G/Ad(H)$. The boundary conditions (2.8) restrict the boundary values of $[g]$ to the projection to $G/Ad(H)$ of the rotated G -group conjugacy class $C_{\mu_n}^G e^{-2\pi i \nu_n}$ (but contain more data if the decomposition $g_nh_n^{-1}$ is not unique). For example, for $G = SU(2)$ with elements $\begin{pmatrix} z & -z' \\ \bar{z}' & \bar{z} \end{pmatrix}$, where $|z|^2 + |z'|^2 = 1$, and for $H = U(1)$, the coset space $G/Ad(H)$ may be identified with the unit disc $D = \{z \mid |z| \leq 1\}$. The boundary conditions (2.8) with (λ_n, ν_n) corresponding to (j_n, m_n) restrict the boundary values of $[g]$ to the intervals $[e^{2\pi i(j-m)/k}, e^{-2\pi i(j+m)/k}] \subset D$ with $2k$ end-points on the disc boundary. Since the conjugacy classes of $U(1)$ are composed of single points, the decomposition $g_nh_n^{-1}$ is unique in this case, given the conjugacy class labels. Imposing the restriction $j = m \bmod 1$ eliminates half of the interval endpoints [25].

3 Canonical structure of the WZW and coset theories

The classical field theory studies the solutions of the variational problem $\delta S = 0$ determined by the action functional S . The space of solutions on a worldsheet with the product structure $\mathbf{R} \times \mathcal{N}$ and Minkowski signature admits a canonical closed 2-form Ω , see e.g. [14] or [18]. If the latter is degenerate (a situation in gauge theories, where the degenerate directions correspond to local gauge transformations), one passes to the space of leaves of the degeneration distribution. By definition, the resulting space is the phase space of the

theory and it carries the canonical symplectic structure⁴.

3.1 Bulk WZW model

Let us start with the well known case of the WZW model on the cylinder $\Sigma = \mathbf{R} \times S^1 = \{(t, x \bmod 2\pi)\}$. The variational equation $\delta S = 0$ becomes a non-linear version of the wave equation

$$\partial_+(g^{-1}\partial_-g) = 0, \quad (3.1)$$

where $\partial_\pm = \partial_{x^\pm}$ with $x^\pm = x \pm t$. The solutions may be labeled by the Cauchy data $g(t, \cdot)$ and $(g^{-1}\partial_t g)(t, \cdot)$. The space of solutions forms the phase space \mathcal{P}^{WZW} of the bulk WZW model. Its canonical symplectic form is given by the expression [14]

$$\Omega^{WZ} = \frac{k}{4\pi} \int_0^{2\pi} \text{tr} \left[-\delta(g^{-1}\partial_t g) g^{-1}\delta g + 2(g^{-1}\partial_+ g)(g^{-1}\delta g)^2 \right](t, x) dx \quad (3.2)$$

which is t -independent⁵. Similarly as for the wave equation, the general solution of (3.1) may be decomposed as

$$g(t, x) = g_\ell(x^+) g_r(x^-)^{-1}. \quad (3.3)$$

The left-right movers $g_{\ell, r} : \mathbf{R} \rightarrow G$ are not necessarily periodic but satisfy $g_{\ell, r}(y + 2\pi) = g_{\ell, r}(y) \gamma$ for the same $\gamma \in G$. They are determined uniquely up to the simultaneous right multiplication by an element of G . The expression of the symplectic form in terms of the left-right movers is described in Appendix A. The currents

$$J_\ell = ik g \partial_+ g^{-1} = ik g_\ell \partial_+ g_\ell^{-1}, \quad J_r = ik g^{-1} \partial_- g = ik g_r \partial_+ g_r^{-1} \quad (3.4)$$

generate the current algebra symmetries of the theory. The conformal symmetries are generated by the components

$$T_\ell = \frac{1}{2k} \text{tr} J_\ell^2, \quad T_r = \frac{1}{2k} \text{tr} J_r^2 \quad (3.5)$$

of the energy momentum tensor.

3.2 Bulk G/H model

In the same cylindrical worldsheet geometry $\Sigma = \mathbf{R} \times S^1$, the classical equations for the coset G/H model take the form:

$$D_+(g^{-1}D_-g) = 0, \quad E g^{-1}D_-g = 0 = E g D_+g^{-1}, \quad F(A) = 0, \quad (3.6)$$

where $D_\pm = \partial_\pm + [A_\pm, \cdot]$ are the light-cone covariant derivatives, E is the orthogonal projection of \mathfrak{g} onto \mathfrak{h} and $F(A) = dA + A^2$ is the curvature of A . The equations are preserved by the H -valued gauge transformations of the fields. The gauge transformations

⁴We ignore here the eventual problems with the infinite-dimensional character of the spaces and singularities that may be usually dealt with in concrete situations

⁵We use the symbol δ for the exterior derivative on the space of classical solutions

provide for the degeneration of the canonical closed 2-form on the space of solutions so that the phase space $\mathcal{P}^{G/H}$ of the bulk coset theory is composed of the gauge-orbits of the solutions of the classical equations (3.6).

The gauge-orbits of solutions may be parametrized in a more effective way. The flat gauge field A may be expressed as $h^{-1}dh$ for $h : \mathbf{R}^2 \rightarrow H$ such that $h(t, x + 2\pi) = \rho^{-1}h(t, x)$ for some $\rho \in H$. The map h is determined uniquely up to the left multiplication by an element of H . Let us set $\tilde{g} = hgh^{-1}$. Note that $\tilde{g} : \mathbf{R}^2 \rightarrow G$ with $\tilde{g}(t, y + 2\pi) = \rho^{-1}\tilde{g}(t, y)\rho$. In terms of field \tilde{g} , the classical equations reduce to

$$\partial_+(\tilde{g}^{-1}\partial_-\tilde{g}) = 0, \quad E \tilde{g}^{-1}\partial_-\tilde{g} = 0 = E \tilde{g} \partial_+\tilde{g}^{-1}. \quad (3.7)$$

The gauge-orbits of the classical solutions of (3.6) are in one-to-one correspondence with the orbits of pairs (\tilde{g}, ρ) under the simultaneous conjugation by elements of H . In terms of these data, the canonical symplectic form on $\mathcal{P}^{G/H}$, obtained following the general prescriptions of [14], is given by

$$\begin{aligned} \Omega^{G/H} = & \frac{k}{4\pi} \int_0^{2\pi} \text{tr} \left[-\delta(\tilde{g}^{-1}\partial_t\tilde{g}) \tilde{g}^{-1}\delta\tilde{g} + 2(\tilde{g}^{-1}\partial_+\tilde{g})(\tilde{g}^{-1}\delta\tilde{g})^2 \right] (t, x) dx \\ & + \frac{k}{4\pi} \text{tr} \left[(\delta\rho)\rho^{-1} \left(\tilde{g}(t, 0)^{-1}(\delta\rho)\rho^{-1}\tilde{g}(t, 0) - (\tilde{g}^{-1}\delta\tilde{g})(t, 0) - ((\delta\tilde{g})\tilde{g}^{-1})(t, 0) \right) \right] \end{aligned} \quad (3.8)$$

for any fixed t . The solutions of the classical equations (3.7) may be expressed again by the left-right movers: $\tilde{g}(t, x) = g_\ell(x^+)g_r(x^-)^{-1}$, where $g_{\ell,r} : \mathbf{R} \rightarrow G$ are such that

$$E g_{\ell,r} \partial_y g_{\ell,r}^{-1} = 0 \quad \text{and} \quad g_{\ell,r}(y + 2\pi) = \rho^{-1} g_{\ell,r}(y) \gamma. \quad (3.9)$$

Given \tilde{g} , the one-dimensional fields $g_{\ell,r}$ are determined up to the simultaneous right multiplication by an element of G . The expression for the symplectic form $\Omega^{G/H}$ in terms of the left-right movers is given in Appendix A. The left-right components of the energy-momentum tensor

$$\begin{aligned} T_\ell &= -\frac{k}{2} \text{tr} (g D_+ g^{-1})^2 = -\frac{k}{2} \text{tr} (g_\ell \partial_+ g_\ell^{-1})^2, \\ T_r &= -\frac{k}{2} \text{tr} (g^{-1} D_- g)^2 = -\frac{k}{2} \text{tr} (g_r \partial_+ g_r^{-1})^2 \end{aligned} \quad (3.10)$$

generate the conformal symmetries of the bulk coset model.

3.3 Bulk G/G model

For the topological coset G/G theory, the classical equations (3.7) reduce to $\tilde{g}^{-1}d\tilde{g} = 0$, i.e. \tilde{g} is constant and it commutes with the monodromy ρ . The phase space $\mathcal{P}^{G/G}$ may be identified with the space of commuting pairs (\tilde{g}, ρ) in G modulo simultaneous conjugations. It is finite-dimensional, in agreement with the topological character of the theory. It comes equipped with the symplectic form

$$\Omega^{G/G} = \frac{k}{4\pi} \text{tr} \left[(\delta\rho)\rho^{-1} \left(\tilde{g}^{-1}(\delta\rho)\rho^{-1}\tilde{g} - \tilde{g}^{-1}\delta\tilde{g} - (\delta\tilde{g})\tilde{g}^{-1} \right) \right]. \quad (3.11)$$

Up to a simultaneous conjugation, $\tilde{g} = e^{2\pi i \mu}$ and $\rho = e^{2\pi i \nu}$ for μ and ν in the Cartan algebra and the symplectic form becomes a constant form on the product of two copies of the Cartan algebra.

$$\Omega^{G/G} = 2\pi k \operatorname{tr} [d\nu d\mu]. \quad (3.12)$$

In particular, conjugation-invariant functions of \tilde{g} Poisson-commute and so do those of ρ .

3.4 Boundary WZW model

The canonical treatment of the boundary theories is quite analogous to that of the bulk ones, except for the necessity to treat the boundary contributions. We consider the strip geometry $\Sigma = \mathbf{R} \times [0, \pi]$ with Minkowski signature and impose on the field $g : \Sigma \rightarrow G$ of the WZW model the boundary conditions discussed in Sect. 2:

$$g(t, 0) \in C_{\mu_0}, \quad g(t, \pi) \in C_{\mu_\pi}. \quad (3.13)$$

For variations δg tangent to the space of fields respecting conditions (3.13), the classical equations $\delta S(g) = 0$ reduce to the bulk equation (3.1) supplemented with the boundary equations

$$g^{-1} \partial_- g + g \partial_+ g^{-1} = 0 \quad \text{for } x = 0, \pi. \quad (3.14)$$

The classical solutions obeying (3.13) form the phase space $\mathcal{P}_{\mu_0 \mu_\pi}^{WZ}$ of the boundary WZW model. Its symplectic form is given by [18]

$$\begin{aligned} \Omega_{\mu_0 \mu_\pi}^{WZ} = & \frac{k}{4\pi} \int_0^\pi \operatorname{tr} \left[-\delta(g^{-1} \partial_t g) g^{-1} \delta g + 2(g^{-1} \partial_+ g) (g^{-1} \delta g)^2 \right] (t, x) dx \\ & + \frac{k}{4\pi} \left[\omega_{\mu_0}(g(t, 0)) - \omega_{\mu_\pi}(g(t, \pi)) \right] \end{aligned} \quad (3.15)$$

for any fixed t . As in the bulk, the classical equations may be solved explicitly, as was described in [18]. We have⁶

$$g(t, x) = g_\ell(x^+) m_0 g_\ell(-x^-)^{-1} = g_\ell(x^+) m_\pi g_\ell(2\pi - x^-)^{-1}, \quad (3.16)$$

where $m_0 \in C_{\mu_0}$, $m_\pi \in C_{\mu_\pi}$ and $g_\ell : \mathbf{R} \rightarrow G$ satisfy

$$g_\ell(y + 2\pi) = g_\ell(y) \gamma \quad \text{for} \quad \gamma = m_0^{-1} m_\pi. \quad (3.17)$$

Note that the boundary conditions (3.13) are fulfilled. The orbits of (g_ℓ, m_0, m_π) under the right multiplication of g_ℓ by elements of G accompanied by the inverse adjoint action on m_0 and m_π are in one-to-one correspondence with the classical solutions. The expression of the symplectic form in terms of these data is given in Appendix A. The boundary WZW theory has a single current $J = ik g_\ell \partial_+ g_\ell^{-1}$ with the corresponding energy-momentum tensor $T = \frac{1}{2k} \operatorname{tr} J^2$.

⁶We use here a slightly different parametrization of the solutions than in [18].

3.5 Boundary G/H model

For the boundary coset G/H model with the G -valued field g and $i\mathbf{h}$ -valued gauge-field A defined on the strip $\mathbf{R} \times [0, \pi]$, we shall impose the boundary conditions

$$g(t, 0) = g_0(t) h_0(t)^{-1}, \quad g(t, \pi) = g_\pi(t) h_\pi(t)^{-1} \quad (3.18)$$

with g_0 , h_0 , g_π and h_π mapping the boundary lines into the conjugacy classes $C_{\mu_0}^G$, $C_{\nu_0}^H$, $C_{\mu_\pi}^G$ and $C_{\nu_\pi}^H$, respectively, see (2.8). The gauge fields A will not be restricted. We shall label such boundary conditions by the pairs (M_0, M_π) , where $M_0 = (\mu_0, \nu_0)$ and $M_\pi = (\mu_\pi, \nu_\pi)$. The variational equations $\delta S(g, A) = 0$ reduce now to the bulk equations (3.6) supplemented with the boundary equations

$$h_0^{-1} D_t h_0 = 0 = h_\pi^{-1} D_t h_\pi, \quad (3.19)$$

$$(g^{-1} D_- g)(\cdot, 0) + h_0 (g D_+ g^{-1})(\cdot, 0) h_0^{-1} = 0, \quad (3.20)$$

$$(g^{-1} D_- g)(\cdot, \pi) + h_\pi (g D_+ g^{-1})(\cdot, \pi) h_\pi^{-1} = 0, \quad (3.21)$$

where $D_t = D_+ - D_-$ is the covariant derivative along the boundary.

The flat gauge field A may be gauged away by representing it as $h^{-1}dh$ for h mapping the strip to G . Setting as in the bulk geometry $\tilde{g} = hgh^{-1}$ and, on the boundary components, $\tilde{g}_0 = hg_0h^{-1}$, $\tilde{h}_0 = hh_0h^{-1}$, and similarly for \tilde{g}_π and \tilde{h}_π , we reduce the bulk equations to (3.7) and the boundary equations to

$$(\tilde{g}^{-1} \partial_- \tilde{g})(\cdot, 0) + n_0 (\tilde{g} \partial_+ \tilde{g}^{-1})(\cdot, 0) n_0^{-1} = 0, \quad (3.22)$$

$$(\tilde{g}^{-1} \partial_- \tilde{g})(\cdot, \pi) + n_\pi (\tilde{g} \partial_+ \tilde{g}^{-1})(\cdot, \pi) n_\pi^{-1} = 0, \quad (3.23)$$

with \tilde{h}_0 and \tilde{h}_π equal, respectively, to constant elements $n_0 \in C_{\nu_0}^H$ and $n_\pi \in C_{\nu_\pi}^H$. The boundary conditions (3.18) become:

$$\tilde{g}(t, 0) = \tilde{g}_0(t) n_0^{-1} \quad \tilde{g}(t, \pi) = \tilde{g}_\pi(t) n_\pi^{-1} \quad (3.24)$$

for \tilde{g}_0 mapping the line into $C_{\mu_0}^G$ and \tilde{g}_π into $C_{\mu_\pi}^G$. As in the bulk case, the phase space $\mathcal{P}_{M_0 M_\pi}^{G/H}$ of the G/H coset model with the boundary conditions (3.18) is composed of the gauge-orbits of the classical solutions. The latter are in one-to-one correspondence with the orbits of the triples (\tilde{g}, n_0, n_π) under the simultaneous conjugation by elements of H . In this parametrization, the symplectic form of the boundary theory is given by

$$\begin{aligned} \Omega_{M_0 M_\pi}^{G/H} &= \frac{k}{4\pi i} \int_0^\pi \text{tr} \left[-\delta(\tilde{g}^{-1} \partial_t \tilde{g}) \tilde{g}^{-1} \delta \tilde{g} + 2(\tilde{g}^{-1} \partial_+ \tilde{g}) (\tilde{g}^{-1} \delta \tilde{g})^2 \right] (t, x) dx \\ &\quad + \frac{k}{4\pi i} \left[\omega_{\mu_0}^G(\tilde{g}_0(t)) - \omega_{\nu_0}^H(n_0) + \text{tr} (\tilde{g}_0^{-1} \delta \tilde{g}_0)(t) n_0^{-1} \delta n_0 \right] \\ &\quad - \frac{k}{4\pi i} \left[\omega_{\mu_\pi}^G(\tilde{g}_\pi(t)) - \omega_{\nu_\pi}^H(n_\pi) + \text{tr} (\tilde{g}_\pi^{-1} \delta \tilde{g}_\pi)(t) n_\pi^{-1} \delta n_\pi \right] \end{aligned} \quad (3.25)$$

for any fixed t .

Similarly as in the WZW case, see (3.16) and (3.17), the two-dimensional fields \tilde{g} satisfying the classical equations (3.22), (3.23) and the boundary conditions (3.24) may be rewritten in terms of a one-dimensional field g_ℓ as

$$\tilde{g}(t, x) = g_\ell(x^+) m_0 g_\ell(-x^-)^{-1} n_0^{-1} = g_\ell(x^+) m_\pi g_\ell(2\pi - x^-)^{-1} n_\pi^{-1} \quad (3.26)$$

with $m_0 \in C_{\mu_0}^G$, $m_\pi \in C_{\mu_\pi}^G$ and $g_\ell : \mathbf{R} \rightarrow G$ satisfying

$$E g_\ell \partial_y g_\ell^{-1} = 0 \quad \text{and} \quad g_\ell(y + 2\pi) = \rho^{-1} g_\ell(y) \gamma \quad \text{for} \quad \rho = n_0^{-1} n_\pi, \quad \gamma = m_0^{-1} m_\pi. \quad (3.27)$$

Given (\tilde{g}, n_0, n_π) , the triple (g_ℓ, m_0, m_π) is determined up to the right multiplication of g_ℓ by an element of G accompanied by the adjoint action of its inverse on m_0 and m_π . The symplectic form $\Omega_{M_0 M_\pi}^{G/H}$ may be rewritten in terms of the data $(g_\ell, m_0, n_0, m_\pi, m_0, n_\pi)$. The result is given by formula (A.2) in Appendix A. The single energy-momentum component of the boundary G/H model is $T = -\frac{k}{2} \text{tr}(g_\ell \partial_+ g_\ell^{-1})^2$.

3.6 Boundary G/G model

In the boundary topological coset G/G theory, the classical equations imply that $\tilde{g}_0, \tilde{h}_0, \tilde{g}_\pi, \tilde{h}_\pi$ and $\tilde{g} = \tilde{g}_0 \tilde{h}_0^{-1} = \tilde{g}_\pi \tilde{h}_\pi^{-1}$ are all constant so that the phase space $\mathcal{P}_{M_0 M_\pi}^{G/G}$ of the boundary G/G theory is composed of the orbits under simultaneous conjugations of the quadruples

$$(\tilde{g}_0, \tilde{h}_0, \tilde{g}_\pi, \tilde{h}_\pi) \in C_{\mu_0} \times C_{\nu_0} \times C_{\mu_\pi} \times C_{\nu_\pi} \subset G^4 \quad \text{with} \quad \tilde{g}_0 \tilde{h}_0^{-1} = \tilde{g}_\pi \tilde{h}_\pi^{-1}. \quad (3.28)$$

The symplectic form is given by

$$\begin{aligned} \Omega_{M_0 M_\pi}^{G/G} &= \frac{k}{4\pi i} \left[\omega_{\mu_0}(\tilde{g}_0) - \omega_{\nu_0}(\tilde{h}_0) + \text{tr}(\tilde{g}_0^{-1} \delta \tilde{g}_0) \tilde{h}_0^{-1} \delta \tilde{h}_0 \right] \\ &\quad - \frac{k}{4\pi i} \left[\omega_{\mu_\pi}(\tilde{g}_\pi) - \omega_{\nu_\pi}(\tilde{h}_\pi) + \text{tr}(\tilde{g}_\pi^{-1} \delta \tilde{g}_\pi) \tilde{h}_\pi^{-1} \delta \tilde{h}_\pi \right]. \end{aligned} \quad (3.29)$$

Using this expression, one may check that conjugation invariant functions of \tilde{g} Poisson commute. Below, we shall find a more transparent interpretation for the phase spaces of the two-dimensional theories described above, including the last example.

4 Canonical structure of the CS theory

The classical Chern-Simons theory [31, 34] is determined by the action functional of \mathfrak{g} -valued 1-forms \mathcal{A} (gauge fields, connections) on an oriented 3-manifold \mathcal{M}

$$S^{CS}(\mathcal{A}) = -\frac{k}{4\pi} \int_{\mathcal{M}} \text{tr} \left[\mathcal{A} d\mathcal{A} + \frac{2}{3} \mathcal{A}^3 \right] \quad (4.1)$$

that does not require a metric on \mathcal{M} for its definition.

4.1 Case without boundary

If \mathcal{M} has no boundary then, under the G -valued gauge transformations $g : \mathcal{M} \rightarrow G$

$$S^{CS}(gAg^{-1} + g dg^{-1}) = S^{CS}(\mathcal{A}) - \frac{k}{4\pi} \int_{\mathcal{M}} g^* \chi. \quad (4.2)$$

In particular, the action is invariant under gauge transformations homotopic to 1 and, for integer k , $e^{-S^{CS}(A)}$ is invariant under all gauge transformations. The classical equations $\delta S^{CS} = 0$ are well known to require that $F(\mathcal{A}) = 0$ with the solutions corresponding to flat connections. In the cylindrical geometry $\mathcal{M} = \mathbf{R} \times \Sigma$ with $\partial\Sigma = \emptyset$, the canonical closed 2-form on the space of solutions is degenerate along the gauge directions. Writing $\mathcal{A} = A + A_0 dt$ where A is tangent to Σ and t is the coordinate of \mathbf{R} , we may use the gauge freedom to impose the condition $A_0 = 0$. In this gauge, the classical equations reduce to

$$\partial_t A = 0, \quad F(A) = 0 \quad (4.3)$$

with the solutions given by static flat connections on the surface Σ . The canonical closed 2-form on the space of solutions becomes

$$\Omega^{CS} = \frac{k}{4\pi} \int_{\Sigma} \text{tr}(\delta A)^2. \quad (4.4)$$

Its degeneration is given by the static gauge transformations $A \mapsto gAg^{-1} + g dg^{-1}$. The phase space \mathcal{P}^{CS} of the CS theory is then composed of the gauge-orbits of flat gauge fields A . In other words, \mathcal{P}^{CS} coincides with the moduli space of flat connections on Σ . Formula (4.4) defines the canonical symplectic structure on \mathcal{P}^{CS} . Below, we shall need several refinements of the above well known scheme.

4.2 Wilson lines

First of all, the CS theory may be coupled to a Wilson line $\mathcal{C} \subset \mathcal{M}$ marked with a label μ belonging to the Cartan subalgebra of \mathfrak{g} . Let γ be a G -valued map defined on the line \mathcal{C} . In the presence of these data, the action functional is modified to

$$S^{CS}(\mathcal{A}, \gamma) = S^{CS}(\mathcal{A}) + ik \int_{\mathcal{C}} \text{tr} \mu \gamma^{-1} (d + \mathcal{A}) \gamma = S^{CS}(gAg^{-1} + g dg^{-1}, g\gamma) \quad (4.5)$$

for g homotopic to 1. The corresponding classical equations read

$$F(\mathcal{A}) = 2\pi i \gamma \mu \gamma^{-1} \mathcal{C}, \quad \left(d(\gamma \mu \gamma^{-1}) + [\mathcal{A}, \gamma \mu \gamma^{-1}] \right) \mathcal{C} = 0, \quad (4.6)$$

where \mathcal{C} is viewed as a singular current. They imply that \mathcal{A} is a flat connection with a singularity on \mathcal{C} . In the cylindrical geometry $\mathcal{M} = \mathbf{R} \times \Sigma$ with the Wilson line $\mathbf{R} \times \{\xi\}$ we may still go to the $A_0 = 0$ gauge in which the classical equations reduce to

$$\partial_t A = 0, \quad \partial_t(\gamma \mu \gamma^{-1}) = 0, \quad F(A) = 2\pi i \gamma \mu \gamma^{-1} \delta_{\xi}. \quad (4.7)$$

The canonical 2-form Ω_{μ}^{CS} on the space of solutions has now the form

$$\Omega_{\mu}^{CS} = \Omega^{CS} - ik \text{tr} \mu (\gamma^{-1} d\gamma)^2 \quad (4.8)$$

where the last term is the Kirillov-Kostant symplectic form on the (co)adjoint orbit \mathcal{O}_μ in \mathfrak{g} passing through μ . The orbits of pairs $(A, \gamma\mu\gamma^{-1})$ solving (4.7) under the time-independent gauge transformations⁷ form the phase space \mathcal{P}_μ^{CS} of the theory. Ω_μ^{CS} defines on \mathcal{P}_μ^{CS} the canonical symplectic structure. Of course, one may consider the CS theory with several Wilson lines.

4.3 Boundaries

If the 3-manifold \mathcal{M} has a boundary then one needs to impose boundary conditions on the gauge fields \mathcal{A} . In the cylindrical geometry $\mathcal{M} = \mathbf{R} \times \Sigma$ where $\partial\Sigma \neq \emptyset$, we may require that

$$A_0 = 0 \quad \text{on} \quad \mathbf{R} \times \partial\Sigma. \quad (4.9)$$

The classical equations are still given by $F(\mathcal{A}) = 0$ and the closed 2-form by (4.4). The only modification is that the degeneration of the latter is given now by the gauge transformations equal to 1 on $\partial\Sigma$. The same remarks pertain to the case with time-like Wilson lines.

4.4 Double CS theory

The last modification of the CS theory on a 3-manifold $\mathcal{M} = \mathbf{R} \times \Sigma$ that we shall need is the double theory [28] with a pair $(\mathcal{A}, \mathcal{B})$ of the, respectively, group G and group $H \subset G$ gauge fields. The action functional of the double theory is the difference of the CS actions for group G and H :

$$S^{2CS}(\mathcal{A}, \mathcal{B}) = S^S(\mathcal{A}) - S^{CS}(\mathcal{B}). \quad (4.10)$$

On the boundary $\mathbf{R} \times \partial\Sigma$ we shall impose the boundary conditions

$$(1 - E) A_0 = 0, \quad E A_0 = B_0, \quad E A_\tau = B_\tau, \quad (4.11)$$

where A_τ denotes the component of A tangent to $\partial\Sigma$. The phase space of the double theory \mathcal{P}^{2CS} is composed of the pairs (A, B) of flat connections on Σ satisfying the last condition of (4.11), modulo G -valued gauge transformations of A and H -valued ones of B that coincide on the boundary of Σ . The symplectic form

$$\Omega^{2CS} = \frac{k}{4\pi} \int_{\Sigma} \text{tr} \left[(\delta A)^2 - (\delta B)^2 \right]. \quad (4.12)$$

Clearly, both gauge fields may be coupled to time-like Wilson lines with labels in the Cartan subalgebras of \mathfrak{g} and \mathfrak{h} , respectively. In the particular case when $H = G$, the double CS theory reduces to the single one on the space $\mathbf{R} \times \tilde{\Sigma}$ with the double surface $\tilde{\Sigma} = \Sigma \# (-\Sigma)$ obtained by gluing Σ along the boundary to its copy with reversed orientation. The phase spaces reduce accordingly.

⁷In fact, the singular terms in (4.7) require some care. A possible way is to consider only solutions of (4.7) that around ξ are of the form $A = i\gamma\mu\gamma^{-1}d\varphi$, where φ denotes the argument of a local complex parameter and to admit the gauge transformations that are constant around ξ . Different choices of local parameters lead then to canonically isomorphic phases spaces.

5 Symplectic relations between the WZW, coset and CS theories

The symplectic structure of the phase spaces of the WZW and coset theories is given by the complicated expressions, see (3.2), (3.8), (3.11), (3.15), (3.25), (3.29), (A.1) and (A.2). Although obtained by applying the general procedures of [14][18], these expressions are far from being transparent. On the other hand, the interpretation of the symplectic structure of the phase spaces of the CS theory determined by the standard constant symplectic form on the space of two-dimensional gauge fields and by the Kirillov-Kostant form on the coadjoint orbits, see (4.4), (4.8) or (4.12), has a clear interpretation. In the present section, we shall describe symplectic isomorphisms between the phase spaces of the WZW and coset theories and those of the CS theory, elucidating this way the canonical structure of the first ones. The existence of such isomorphisms for the bulk WZW and coset theories has been known for long time, see [34][8][28]. We only give their slightly more explicit realization. The isomorphism of the boundary WZW theory phase space with a moduli space of flat connections on a twice punctured disc has been first described in [18]. It represents another aspect of the relations between the boundary conformal theories and the topological three-dimensional theories developed in [11][12].

5.1 Bulk WZW model

The bulk WZW model on $\mathbf{R} \times S^1$ corresponds to the CS theory on $\mathbf{R} \times \mathcal{Z}$, where $\mathcal{Z} = \{z \mid \frac{1}{2} \leq |z| \leq 1\}$, with the boundary condition $A_0 = 0$. The isomorphism I between the phase spaces \mathcal{P}^{CS} and \mathcal{P}^{WZ} of the two theories is defined by the formula giving the classical solution of the WZW model on $\mathbf{R} \times S^1$ in terms of a flat connection on \mathcal{Z} :

$$g(t, x) = P e^{\int_{\ell_{x,t}} A}, \quad (5.1)$$

where $P e^{\int_{\ell} \cdots}$ stands for the path-ordered (from left to right) exponential and $\ell_{x,t}$ is an appropriate contour, see Fig. 2.

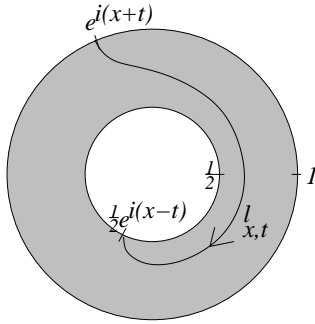


Fig. 2

In particular, $\ell_{x,0}$ is a radial segment from e^{ix} to $\frac{1}{2}e^{ix}$ and $\ell_{x,t}$ is obtained from $\ell_{x,0}$ by rotating continuously the beginning of the segment by angle t and its end by angle $-t$. It is not difficult to see that I is a symplectic isomorphism, see Appendix B. In terms of the CS gauge field A , the currents (3.4) become $J_\ell(x^+) = ikA_\varphi(e^{ix^+})$ and $J_r(x^-) = -ikA_\varphi(\frac{1}{2}e^{ix^-})$, where A_φ denotes the angular component of A .

5.2 Bulk G/H model

For the bulk G/H coset model, the corresponding CS theory is the double one on $\mathbf{R} \times \mathcal{Z}$. Recall that the phase space of the latter is formed by the gauge-orbits of pairs (A, B) of, respectively, group G and group H flat connections on \mathcal{Z} whose components tangent to the boundary are related by $EA_\varphi = B_\varphi$. Choose a base point $1 \in \mathcal{Z}$. We may consider $w = \frac{1}{i} \ln z = \int_1^z \frac{dz}{iz}$ as the coordinate on the covering space $\tilde{\mathcal{Z}}$ of \mathcal{Z} . Let us introduce two maps g_A and h_B from $\tilde{\mathcal{Z}}$ to G and H , respectively,

$$g_A(w) = P e^{\int_1^w A}, \quad h_B(w) = P e^{\int_1^w B}. \quad (5.2)$$

Clearly $A = g_A dg_A^{-1}$ and $B = h_B dh_B^{-1}$. We shall set

$$\tilde{g}(t, x) = h_B(x+t)^{-1} g_A(x+t) g_A(x-t+w_0)^{-1} h_B(x-t+w_0) \quad (5.3)$$

for $w_0 = i \ln 2$. Note that $\tilde{g}(t, x+2\pi) = \rho^{-1} \tilde{g}(t, x) \rho$ where

$$\rho = P e^{\int_C B} \quad (5.4)$$

with C the clock-wise contour around the unit circle from 1 to 1. It is straightforward to check that \tilde{g} satisfies the classical equations (3.7) of the bulk G/H coset model. Under the G -valued gauge transformations of A and H -valued ones of B that agree on the boundary of \mathcal{Z} , the pair (\tilde{g}, ρ) undergoes a simultaneous conjugation by a fixed element of H . We infer that (5.3) defines an injective map I' from the phase space \mathcal{P}^{2CS} of the double CS theory on $\mathbf{R} \times \mathcal{Z}$ to the phase space $\mathcal{P}^{G/H}$ of the bulk coset G/H model. Using the parametrization of the solutions \tilde{g} of the coset model by the left-right movers

$$g_\ell(y) = h_B^{-1}(y) g_A(y), \quad g_r(y) = h_B(y+w_0)^{-1} g_A(y+w_0), \quad (5.5)$$

satisfying (3.9) for ρ given by (5.4) and

$$\gamma = P e^{\int_C B}, \quad (5.6)$$

it is easy to see that the map I' is also onto. The main result is that it defines a symplectic isomorphism, see Appendix B.

5.3 Bulk G/G model

In the special case of $H = G$ where the phase space \mathcal{P}^{2CS} reduces to that of the gauge-orbits of flat connections on the torus represented as the double surface $\mathcal{Z} \# (-\mathcal{Z})$, the field

\tilde{g} of (5.3) is (t, x) -independent and it describes the parallel transport around the a -cycle, see Fig. 3.

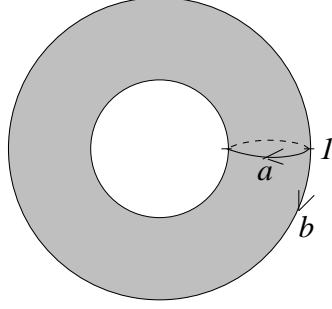


Fig. 3

Similarly, the monodromy ρ , commuting with \tilde{g} , describes the parallel transport around the b -cycle. Equations (3.11) and (3.12) express then the symplectic form (4.4) in terms of the holonomy of the gauge field and is a special case of the result of [2]. Note that conjugation-invariant functions of the holonomy around a fixed cycle on the torus $\mathcal{Z}\#(-\mathcal{Z})$ Poisson-commute.

5.4 Boundary WZW model

The case of the boundary WZW model with the boundary conditions (3.13) has been analyzed in [18]. The corresponding CS theory is the one on the solid cylinder $\mathbf{R} \times D$ where D is the unit disc in the complex plane, with two time-like Wilson lines, say $\mathbf{R} \times \{\frac{1}{2}\}$ with label μ_0 and $\mathbf{R} \times \{-\frac{1}{2}\}$ with label $-\mu_\pi$, see Fig. 4.

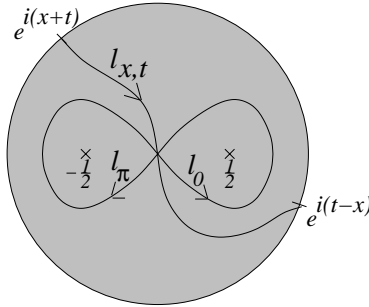


Fig. 4

Let A be a flat connection on D with

$$F(A) = 2\pi i \gamma_0 \mu_0 \gamma_0^{-1} \delta_{\frac{1}{2}} - 2\pi i \gamma_\pi \mu_\pi \gamma_\pi^{-1} \delta_{-\frac{1}{2}}. \quad (5.7)$$

Its holonomy around the contour ℓ_0 of Fig. 4 lies then in the conjugacy class \mathcal{C}_{μ_0} and around ℓ_π in \mathcal{C}_{μ_π} . To each such connection, we may associate a classical solution of the WZW theory on a strip $\mathbf{R} \times [0, \pi]$ by setting

$$g(t, x) = P e^{\int_{\ell_{x,t}} A}, \quad (5.8)$$

with the contour $\ell_{x,t}$ as in Fig. 4. In particular, for $t = 0$, $\ell_{x,0}$ goes from e^{ix} to e^{-ix} crossing once the interval $(-\frac{1}{2}, \frac{1}{2})$. For other times, $\ell_{x,t}$ is obtained by rotating both ends of $\ell_{x,0}$ by the angle t . Note how the boundary conditions (3.13) are assured. The bulk equations (3.1) and the boundary ones (3.14) are also satisfied. The right hand side of (5.8) is clearly invariant under the gauge transformations of A equal to 1 on the boundary of the disc. The decomposition (3.16) of the solution in terms of the one-dimensional field g_ℓ is obtained by setting

$$g_\ell(y) = P e^{\int_{\ell_y} A}, \quad m_0 = P e^{\int_{\ell_0} A}, \quad m_\pi = P e^{\int_{\ell_\pi} A}. \quad (5.9)$$

Here for $y \in [0, \pi]$, the contour ℓ_y coincides with the interval $[e^{iy}, 0]$ and it is deformed continuously for other values of y . Contours ℓ_0 and ℓ_π are as in Fig. 4. We obtain this way an isomorphism $I_{\mu_0\mu_\pi}$ from the phase space $\mathcal{P}_{\mu_0(-\mu_\pi)}^{CS}$ of the CS theory on $\mathbf{R} \times D$ with two time-like Wilson lines onto the phase space $\mathcal{P}_{\mu_0\mu_\pi}^{WZW}$ of the boundary WZW theory. As was explained in [18], $I_{\mu_0\mu_\pi}$ preserves the symplectic structure. We sketch in Appendix B the idea of the proof. In [18], this result was used to quantize the boundary WZW theory.

5.5 Boundary G/H model

Finally, let us consider the coset G/H theory on the strip $\mathbf{R} \times [0, \pi]$ with the (M_0, M_1) boundary conditions (3.18). It corresponds to the double CS theory on $\mathbf{R} \times D$ coupled to Wilson lines. The group G gauge field is coupled to lines $\mathbf{R} \times \{\frac{1}{2}\}$ and $\mathbf{R} \times \{-\frac{1}{2}\}$ with labels μ_0 and $-\mu_\pi$ and the group H gauge field to the same lines with labels ν_0 and $-\nu_\pi$, respectively. Let us define

$$g_A(y) = P e^{\int_{\ell_y} A}, \quad h_B(y) = P e^{\int_{\ell_y} B}, \quad (5.10)$$

with the contour ℓ_y as in (5.9), and

$$m_0 = P e^{\int_{\ell_0} A}, \quad m_\pi = P e^{\int_{\ell_\pi} A}, \quad n_0 = P e^{\int_{\ell_0} B}, \quad n_\pi = P e^{\int_{\ell_\pi} B}. \quad (5.11)$$

The monodromy of g_A and h_B is given by:

$$\begin{aligned} g_A(y + 2\pi) &= g_A(y) \gamma & \text{for } \gamma &= m_0^{-1} m_\pi, \\ h_B(y + 2\pi) &= h_B(y) \rho & \text{for } \rho &= n_0^{-1} n_\pi. \end{aligned} \quad (5.12)$$

Setting $g_\ell(y) = h_B(y)^{-1} g_A(y)$ we obtain a one-dimensional field satisfying (3.27) and describing via (3.26) a classical solution $\tilde{g}(t, x)$ of the boundary G/H coset theory with the

(M_0, M_π) boundary conditions. Clearly, \tilde{g} is invariant under the gauge transformations of A and B equal on the boundary of the disc. We obtain this way an isomorphism $I'_{M_0 M_\pi}$ between the phase space $\mathcal{P}_{M_0(-M_\pi)}^{2CS}$ of the double CS theory on $\mathbf{R} \times D$ with two pairs of Wilson lines and the phase space $\mathcal{P}_{M_0 M_\pi}^{G/H}$ of the boundary coset model. The proof that $I'_{M_0 M_\pi}$ preserves the symplectic structure is similar to the one in the case of the boundary WZW model, see Appendix B.

5.6 Boundary G/G model

In the special case $H = G$, the phase space of the double CS theory reduces to that of the single theory on the 2-sphere $S^2 = D \# (-D)$ with four Wilson lines: $\mathbf{R} \times \{\frac{1}{2}\}$ and $\mathbf{R} \times \{-\frac{1}{2}\}$ in $\mathbf{R} \times D$ with labels μ_0 and $-\mu_\pi$ and their images in $\mathbf{R} \times (-D)$ with labels $-\nu_0$ and ν_π . The group elements and $\tilde{h}_0 \in C_{\nu_0}$, $\tilde{h}_\pi \in C_{\nu_\pi}$ and $\tilde{g} = \tilde{g}_0 \tilde{h}_0^{-1} = \tilde{g}_\pi \tilde{h}_\pi^{-1}$, see (3.28), are given by the contour integrals

$$\tilde{h}_0 = P e^{\int_{\ell'_0} A}, \quad \tilde{h}_\pi = P e^{\int_{\ell'_\pi} A}, \quad \tilde{g} = P e^{\int_{\ell} A}, \quad (5.13)$$

where ℓ'_0 and ℓ'_π are the copies in $-D$ of ℓ_0 and ℓ_π , see Fig. 4, and ℓ is the closed contour as in Fig. 5 starting and ending at the center of $-D$, with the broken pieces contained in $-D$ and the solid ones in D .

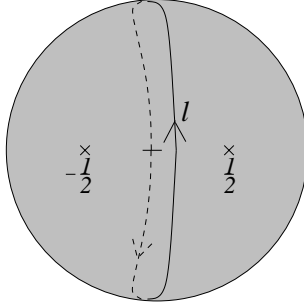


Fig. 5

The equality of the symplectic form on the moduli space of flat connections A on S^2 with four punctures to the form of (3.29) is essentially again a special case of the result of [2].

6 Quantization of the boundary G/G coset model

The description of the canonical structure of the two-dimensional WZW and coset theories in terms of the moduli spaces of flat connections on surfaces with boundary may be further reduced by the “topological fusion” to the case of a disc with a single insertion and of a closed surface with multiple insertions. This provides a good starting point for the

quantization of the theory. Indeed, quantization of the moduli spaces on a disc with a single puncture is an example of the orbit method in the representation theory [24]. It gives rise to the highest weight representations of the current algebra [1][8]. On the other hand, quantization of the moduli spaces of flat connections on closed surfaces leads to the finite-dimensional spaces of conformal blocks of the WZW theories [34] that may be also viewed as spaces of invariant tensors of the quantum group $\mathcal{U}_q(\mathfrak{g})$ for $q = e^{\frac{i\pi}{k+g^\vee}}$ (g^\vee stands for the dual Coxeter number of G). This was described in detail for the boundary WZW theory with $G = SU(2)$ in [18]. Here we shall carry out the quantization program for the boundary coset theory G/G . Recall that the canonical structure of the G/G model has been described directly in terms of the moduli spaces of flat connections on closed surfaces so that no topological fusion will be needed. As a result, we shall obtain an example of a two-dimensional boundary topological field theory, a structure that has recently attracted some attention [23][26][27].

Let us start from the well known case of the bulk G/G coset model. As explained above, the phase space of the theory coincides with the moduli space of flat connections on the torus $\mathcal{Z}\#(-\mathcal{Z})$. The quantization of this moduli space gives rise to the space of the conformal blocks \mathcal{H} of the the WZW theory on the torus⁸. The space is spanned by the affine characters χ_λ^k of level k of the current algebra $\widehat{\mathfrak{g}}$ associated to Lie algebra \mathfrak{g} or by the characters χ_λ of irreducible representations of group G of integrable highest weights λ restricted to the points⁹ $\widehat{\zeta} \equiv 2\pi \frac{\zeta + \rho_W}{k + g^\vee}$ for ζ running through the integrable weights and ρ_W standing for the Weyl vector. As is well known, the restricted characters induce under pointwise multiplication a commutative ring \mathcal{R}^k , the fusion ring of the WZW theory. This way, the space \mathcal{H} of the conformal blocks on the torus becomes a commutative algebra with unity $\mathcal{H} \cong \mathcal{R}^k \otimes \mathbf{C}$. The unit element is given by the character $\chi_0 \equiv 1$ of the trivial representation. \mathcal{H} may be viewed as the algebra of functions on the discrete set $\{\widehat{\zeta} | \zeta \text{ integrable}\} = \text{Spec}(\mathcal{H})$. The operator of multiplication by the restricted character χ_λ is the quantizations of the function $\chi_\lambda(\tilde{g})$ on the phase space $\mathcal{P}^{G/G}$. We may equip \mathcal{H} with a non-degenerate symmetric bilinear form such that¹⁰ $\langle \chi_\lambda, \chi_{\lambda'} \rangle = \delta_{\lambda\lambda'}$, satisfying

$$\langle 1, 1 \rangle = 1, \quad \langle a_1 a_2, a_3 \rangle = \langle a_1, a_2 a_3 \rangle \quad (6.1)$$

for $a_i \in \mathcal{H}$. These are the data of a two-dimensional bulk topological field theory [4]. Such a theory assigns to each compact, not-necessarily connected, oriented surface Σ with the boundary $\partial\Sigma = \sqcup S_n^1$ a linear functional $\mathcal{I}_\Sigma : \bigotimes_n \mathcal{H} \rightarrow \mathbf{C}$, the amplitude of Σ . This assignment is supposed to have three properties. First, \mathcal{I} is supposed to be multiplicative under disjoint products:

$$\mathcal{I}_{\Sigma_1 \sqcup \Sigma_2} = \mathcal{I}_{\Sigma_1} \otimes \mathcal{I}_{\Sigma_2}. \quad (6.2)$$

Second, it should be covariant with respect to surface homeomorphisms preserving orientation. This means that $\mathcal{I}_\Sigma = \mathcal{I}_\Sigma \pi$ for any permutation π of factors in $\bigotimes_n \mathcal{H}$ within

⁸One way to proceed is to use the natural Kähler structure of the moduli space given by its identification with the moduli space of holomorphic vector bundles, see e.g. [9]

⁹We view the characters as functions on the Cartan algebra identified with functions on the group via the exponential map

¹⁰ $\bar{\lambda}$ denotes the highest weight of the representation of G complex conjugate to the one with the highest weight λ .

the connected components of Σ so that \mathcal{I}_Σ depends only on the collection of the numbers of boundary circles and handles in each component. Third, \mathcal{I} is required to be consistent with the gluing of boundary components:

$$\mathcal{I}_{\Sigma_{nm}} = \mathcal{I}_\Sigma P_{nm} \quad (6.3)$$

if Σ_{nm} is obtained from Σ by gluing together the n -th and the m -th boundary components of opposite orientation. $P_{nm} = \sum_\gamma id \otimes \dots \otimes p_{\widehat{n}} \otimes \dots \otimes p'_{\widehat{m}} \otimes \dots \otimes id$ where

$$P = \sum_\gamma p_\gamma \otimes p'_\gamma \in \mathcal{H} \otimes \mathcal{H} \quad (6.4)$$

is the dual of the bilinear form $\langle \cdot, \cdot \rangle$ (we may take $p_\gamma = \chi_\lambda$, $p'_\gamma = \chi_{\bar{\lambda}}$). It is easy to see that if Σ is connected and has g handles then

$$\mathcal{I}_\Sigma \otimes_n a_n = \left\langle \prod_n a_n, P^g \right\rangle. \quad (6.5)$$

In particular, the surfaces of Fig. 6 (with the orientation inherited from the plane) correspond to the amplitudes

$$a \mapsto \langle a, 1 \rangle, \quad a_1 \otimes a_2 \mapsto \langle a_1, a_2 \rangle, \quad a_1 \otimes a_2 \otimes a_3 \mapsto \langle a_1 a_2, a_3 \rangle. \quad (6.6)$$

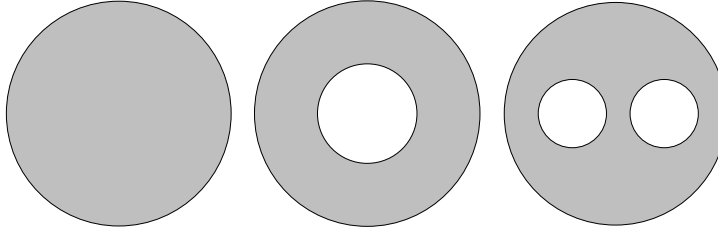


Fig. 6

They permit to reconstruct the amplitudes for all surfaces. For the bulk G/G theory and $a_n = \chi_{\lambda_n}$, expression (6.5) evaluates to an integer $N_{(\lambda_n)}(g)$, the Verlinde dimension of the space of conformal blocks on Σ with insertions of the primary fields with labels λ_n . Explicitly [32],

$$N_{(\lambda_n)}(g) = \sum_\zeta (S_0^\zeta)^{2-2g} \prod_n (S_{\lambda_n}^\zeta / S_0^\zeta), \quad (6.7)$$

where $S_\lambda^\zeta = S_\zeta^\lambda = \overline{S_{\bar{\lambda}}^\zeta}$ are the elements of the matrix giving the modular transformation of the affine characters χ_λ^k . For closed surfaces, \mathcal{I}_Σ is an integer $N(g)$. It is equal to 1 for the sphere and to $\dim(\mathcal{H})$, i.e. the number of integrable weights, for the torus. In short,

the bulk topological G/G coset theory is the theory of the Verlinde dimensions. They may all be obtained from the fusion coefficients $N_{\lambda\eta}^\zeta = N_{\lambda\eta\zeta}(0)$ which define the product in \mathcal{H} : $\chi_\lambda \chi_\eta = \sum N_{\lambda\eta}^\zeta \chi_\zeta$.

We would like to extend this structure to the case of the boundary G/G coset theory with the (M_0, M_π) boundary conditions where $M_0 = (\mu_0, \nu_0)$ and $M_\pi = (\mu_\pi, \nu_\pi)$. Recall that we have identified the phase space $\mathcal{P}_{M_0 M_1}^{G/G}$ of this theory with the moduli space $\mathcal{P}_{M_0(-M_\pi)}^{2CS}$ of flat connections on $S^2 = D \# (-D)$ with four punctures labeled by $\mu_0, -\nu_0, -\mu_\pi$ and ν_π , all in the Cartan algebra of G . For k a positive integer and $\mu_0 = \lambda_0/k, \nu_0 = \eta_0/k, \mu_\pi = \lambda_\pi/k, \nu_\pi = \eta_\pi/k$, where $\lambda_0, \eta_0, \lambda_\pi, \eta_\pi$ are integrable weights, the phase space $\mathcal{P}_{M_0(-M_\pi)}^{2CS}$ gives upon quantization the space of conformal blocks of the WZW theory on $D \# (-D)$ with insertions of the primary fields labeled by λ_0 and $\bar{\lambda}_\pi$ in D and by $\bar{\eta}_0$ and η_π in $-D$. We shall denote this space by $\mathcal{H}_{L_0 L_\pi}$ with $L_0 = (\lambda_0, \eta_0)$ and $L_\pi = (\lambda_\pi, \eta_\pi)$. By the factorization properties of the spaces of conformal blocks,

$$\mathcal{H}_{L_0 L_\pi} \cong \bigoplus_\zeta \text{Hom}(\mathcal{H}_{\lambda_\pi \bar{\eta}_\pi \zeta}, \mathcal{H}_{\lambda_0 \bar{\eta}_0 \zeta}) \quad (6.8)$$

where $\mathcal{H}_{\lambda\eta\zeta}$ denotes the space of conformal blocks on S^2 with insertions of three primary fields labeled by the integrable weights λ, η and ζ . In particular, \mathcal{H}_{LL} is an associative (in general, non-commutative) algebra with unity, a direct sum of matrix algebras. More generally, there is a natural bilinear product $\mathcal{H}_{L_1 L_2} \times \mathcal{H}_{L_2 L_3} \rightarrow \mathcal{H}_{L_1 L_3}$ defined by composition of homomorphisms in each ζ -component. It gives $\mathcal{H}_{L_1 L_2}$ the structure of a left $\mathcal{H}_{L_1 L_1}$ -module and of a right $\mathcal{H}_{L_2 L_2}$ -module. It is useful to consider the direct sum of the boundary spaces $\mathcal{H}_b = \bigoplus_{L_1, L_2} \mathcal{H}_{L_1 L_2}$. The product in \mathcal{H}_b defined by

$$\underline{a} \underline{b} = \left(\sum_L a_{L_1 L} b_{L L_2} \right) \quad (6.9)$$

for $\underline{a} = (a_{L_1 L_2})$ and $\underline{b} = (b_{L_1 L_2})$ makes \mathcal{H}_b an associative algebra with unity $\underline{1} = (\delta_{L_1 L_2})$.

Each space $\mathcal{H}_{L_1 L_2}$ is, additionally, a module of the commutative algebra \mathcal{H} with the character $\chi_\lambda \in \mathcal{R}^k$ acting diagonally in the decomposition (6.8) as the multiplication by $\chi_\lambda(\hat{\zeta})$. This action quantizes the classical observables $\chi_\lambda(\tilde{g})$, where, in the CS description, \tilde{g} is the holonomy around the contour ℓ on Fig. 5, see (5.13). The induced structure of the \mathcal{H} -module on \mathcal{H}_b satisfies

$$a(\underline{b}\underline{c}) = (a\underline{b})\underline{c} = \underline{b}(a\underline{c}) \quad (6.10)$$

for $a \in \mathcal{H}$ and $\underline{b}, \underline{c} \in \mathcal{H}_b$ which is equivalent to the statement that $a\underline{b} = (a\underline{1})\underline{b}$ and that elements $a\underline{1}$ are in the center of \mathcal{H}_b .

We shall equip \mathcal{H}_b with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle_b$ with the only non-vanishing matrix elements between subspaces with permuted boundary labels, i.e. such that

$$\langle \underline{a}, \underline{b} \rangle_b = \sum_{L_1, L_2} \langle a_{L_1 L_2}, b_{L_2 L_1} \rangle_b. \quad (6.11)$$

Explicitly, we shall set:

$$\langle a_{L_1 L_2}, b_{L_2 L_1} \rangle_b = \sum_{\zeta} (\pm S_0^{\zeta}) \operatorname{tr} [a_{L_1 L_2}(\zeta) a_{L_2 L_1}(\zeta)], \quad (6.12)$$

where the sign is fixed once for all. It is easy to see that the bilinear form $\langle \cdot, \cdot \rangle_b$ satisfies

$$\langle \underline{a} \underline{b}, \underline{c} \rangle_b = \langle \underline{a}, \underline{b} \underline{c} \rangle_b. \quad (6.13)$$

The last relation, together with the symmetry of the form implies the cyclic symmetry

$$\langle \underline{a} \underline{b}, \underline{c} \rangle_b = \langle \underline{b} \underline{c}, \underline{a} \rangle_b = \langle \underline{c} \underline{a}, \underline{b} \rangle_b. \quad (6.14)$$

Let

$$P_{L_1 L_2} = \sum_A p_{L_1 L_2}^A \otimes p_{L_2 L_1}^A \in \mathcal{H}_{L_1 L_2} \otimes \mathcal{H}_{L_2 L_1} \quad (6.15)$$

be the dual of the bilinear form (6.12) on $\mathcal{H}_{L_1 L_2} \times \mathcal{H}_{L_2 L_1}$. We may take

$$A = (\zeta, i, j) \quad p_{L_1 L_2}^A = (\pm S_0^{\zeta})^{-\frac{1}{2}} e_{L_1 \zeta}^i e_{L_2 \zeta}^{*j}, \quad p_{L_2 L_1}^A = (\pm S_0^{\zeta})^{-\frac{1}{2}} e_{L_2 \zeta}^j e_{L_1 \zeta}^{*i}, \quad (6.16)$$

where, for $L = (\lambda, \eta)$, $(e_{L\zeta}^i)$ is a basis of $\mathcal{H}_{\lambda\bar{\eta}\zeta}$ and $(e_{L\zeta}^{*i})$ is the dual basis. The bilinear forms on \mathcal{H}_b and on \mathcal{H} are tied together by the relation

$$\sum_A \langle a_{L_1 L_1} p_{L_1 L_2}^A b_{L_2 L_2}, p_{L_2 L_1}^A \rangle_b = \sum_{\gamma} \langle a_{L_1 L_1}, p_{\gamma} \underline{1} \rangle_b \langle p'_{\gamma} \underline{1}, b_{L_2 L_2} \rangle_b. \quad (6.17)$$

Indeed, with the use of (6.16), the left hand side may be rewritten as

$$\sum_{\zeta} \operatorname{tr} [a_{L_1 L_1}(\zeta)] \operatorname{tr} [b_{L_2 L_2}(\zeta)] \quad (6.18)$$

and the right hand side is

$$\sum_{\lambda, \zeta, \zeta'} S_0^{\zeta} \chi_{\lambda}(\hat{\zeta}) S_0^{\zeta'} \chi_{\bar{\lambda}}(\hat{\zeta}') \operatorname{tr} [a_{L_1 L_1}(\zeta)] \operatorname{tr} [b_{L_2 L_2}(\zeta')]. \quad (6.19)$$

Note that the sign ambiguity in the definition (6.12) of the bilinear form on \mathcal{H}_b disappears from both expressions. The equality of the two sides is inferred by using the relations $S_0^{\zeta} \chi_{\lambda}(\hat{\zeta}) = S_{\lambda}^{\zeta}$, $S_0^{\zeta'} \chi_{\bar{\lambda}}(\hat{\zeta}') = \overline{S_{\lambda}^{\zeta'}}$ and the unitarity of the modular matrix (S_{λ}^{ζ}) .

We may abstract from the above construction an algebraic structure

$$(\mathcal{H}, \langle \cdot, \cdot \rangle, \mathcal{H}_b, \langle \cdot, \cdot \rangle_b) \quad (6.20)$$

such that

1. \mathcal{H} is a finite-dimensional associative commutative algebra with unity equipped with the non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$,

2. $\mathcal{H}_b = \oplus \mathcal{H}_{L_1 L_2}$ is a finite-dimensional associative algebra with unity (in general non-commutative) equipped with the non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle_b$,

3. \mathcal{H}_b is an \mathcal{H} -module with each $\mathcal{H}_{L_1 L_2}$ being a submodule,
4. relations (6.1), (6.9), (6.11), (6.13), (6.10) and (6.17) hold.

Such a structure defines a boundary two-dimensional topological field theory [23][26][27]. The amplitudes of such a theory correspond¹¹ to compact oriented surfaces Σ with boundary where the boundary components S_n^1 may contain distinguished closed disjoint subintervals (possibly the whole component) marked with labels L of the boundary conditions, see Fig. 7.

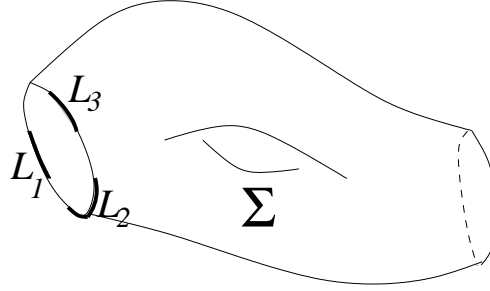


Fig. 7

Let, for each S_n^1 with labeled subintervals, (I_{ns}) be the collection of the remaining subintervals of S_n^1 . The amplitude assigned to such a labeled surface is a linear functional

$$\mathcal{I}_\Sigma : \left(\bigotimes_n' \mathcal{H} \right) \otimes \left(\bigotimes_{n,s}'' \mathcal{H}_b \right) \rightarrow \mathbf{C}, \quad (6.21)$$

where the first tensor product is over the boundary components without labeled subintervals. \mathcal{I}_Σ is required to vanish on all the components $\mathcal{H}_{L_1 L_2}$ of \mathcal{H}_b except those with (L_1, L_2) given by the labels of the intervals adjacent to I_{ns} .

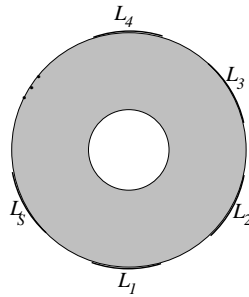


Fig. 8

¹¹There are minor differences between our formulation and that of the above references, mostly a matter of convenience. In particular, we consider only boundary orientations induced from the bulk.

For example, the amplitude of the labeled surface of Fig. 8 is a linear functional on $\mathcal{H} \otimes \mathcal{H}_{L_1 L_2} \otimes \mathcal{H}_{L_2 L_3} \otimes \dots \otimes \mathcal{H}_{L_S L_1}$. The amplitude assignment \mathcal{I} is still required to obey (6.2) and to be covariant under orientation and label preserving homeomorphisms. The latter means that $\mathcal{I}_\Sigma = \mathcal{I}_\Sigma \pi$ for cyclic permutations of boundary intervals and their labels within boundary circles. The amplitudes depend this way on the collections of boundary labels with the cyclic order within each boundary circle (including the empty collection). The consistency with gluing (6.3) is now generalized to include the gluing along two unlabeled intervals of opposite orientation and permuted labels of the adjacent intervals as in Fig. 9. In the latter case the dual bilinear form $P \in \mathcal{H} \otimes \mathcal{H}$ should be replaced in (6.3) by the dual form $P_{L_1 L_2} \in \mathcal{H}_{L_1 L_2} \otimes \mathcal{H}_{L_2 L_1}$ inserted in the appropriate factors of the tensor product $\otimes_{n,s}'' \mathcal{H}_b$.

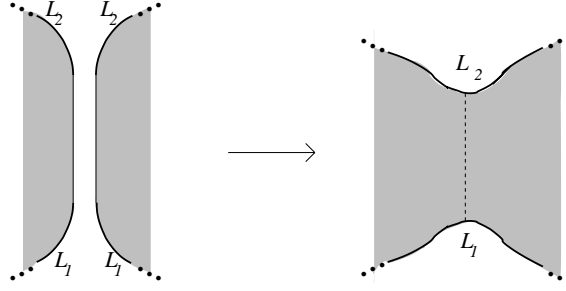


Fig. 9

It is not difficult to construct the amplitudes \mathcal{I}_Σ from the data (6.20). For completeness, we shall describe the argument. First, besides the bulk amplitudes already discussed, it is enough to know only the amplitudes corresponding to labeled surfaces of Fig. 10

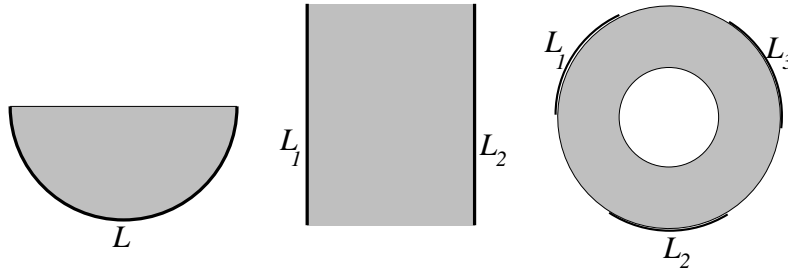


Fig. 10

$$\begin{aligned} \underline{a}_{LL} &\mapsto \langle a_{LL}, \underline{1} \rangle_b, & a_{L_1 L_2} \otimes b_{L_2 L_1} &\mapsto \langle a_{L_1 L_2}, b_{L_2 L_1} \rangle_b, \\ a \otimes a_{L_1 L_2} \otimes b_{L_2 L_3} \otimes c_{L_3 L_1} &\mapsto \langle a_{L_1 L_2} b_{L_2 L_3}, a c_{L_3 L_1} \rangle_b. \end{aligned}$$

Gluing the unlabeled disc to the inner boundary of the annulus gives the disc with three labeled boundary intervals and the amplitude

$$a_{L_1 L_2} \otimes b_{L_2 L_3} \otimes c_{L_3 L_1} \mapsto \langle a_{L_1 L_3} b_{L_3 L_2}, c_{L_2 L_1} \rangle_b. \quad (6.22)$$

We may subsequently glue such a disc to the annulus as in Fig. 11

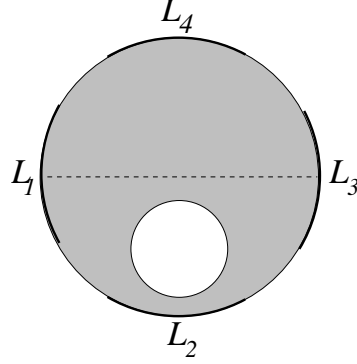


Fig. 11

to obtain the amplitude

$$\begin{aligned} a \otimes a_{L_1 L_2} \otimes b_{L_2 L_3} \otimes c_{L_3 L_4} \otimes d_{L_4 L_1} &\mapsto \sum_A \langle a_{L_1 L_2} b_{L_2 L_3}, a p_{L_3 L_1}^A \rangle_b \langle p_{L_1 L_3}^A c_{L_3 L_4}, d_{L_4 L_1} \rangle_b \\ &= \langle a_{L_1 L_2} b_{L_2 L_3} c_{L_3 L_4}, a d_{L_4 L_1} \rangle_b, \end{aligned} \quad (6.23)$$

where the last equality follows from the trivial identity

$$\sum_A \langle a_{L_1 L_2} p_{L_2 L_1}^A \rangle_b \langle p_{L_1 L_2}^A b_{L_2 L_1} \rangle_b = \langle a_{L_1 L_2}, b_{L_2 L_1} \rangle_b. \quad (6.24)$$

One obtains similarly the amplitudes of the general annuli of Fig. 8. They are given by the linear functionals

$$a \otimes a_{L_1 L_2}^1 \otimes \cdots \otimes a_{L_S L_1}^S \mapsto \langle a_{L_1 L_2}^1 \cdots a_{L_{S-1} L_S}^{S-1}, a a_{L_S L_1}^S \rangle_b \quad (6.25)$$

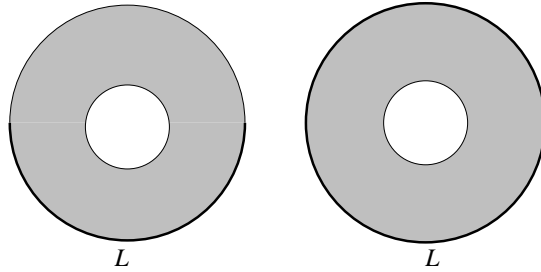


Fig. 12

invariant under the cyclic permutations of $a_{L_1 L_2}^1 \cdots a_{L_S L_1}^S$ due to (6.10) and (6.14). The formula extends to the cases with $S = 1$ and $S = 0$ corresponding to the surfaces depicted in Fig. 12 if we interpret it as giving the linear maps $a \otimes a_{LL} \mapsto \langle a_{LL}, a 1_{LL} \rangle$ and $a \mapsto \langle 1_{LL}, a 1_{LL} \rangle_b$, respectively, where 1_{LL} stands for the unity of \mathcal{H}_{LL} . For a general surface, we may obtain its amplitude by first cutting off the labeled boundary circles around nearby unlabeled ones as in Fig. 13, and then composing the amplitude from that of the annuli of Fig. 8 and of the ones for the surface with unlabeled boundary.

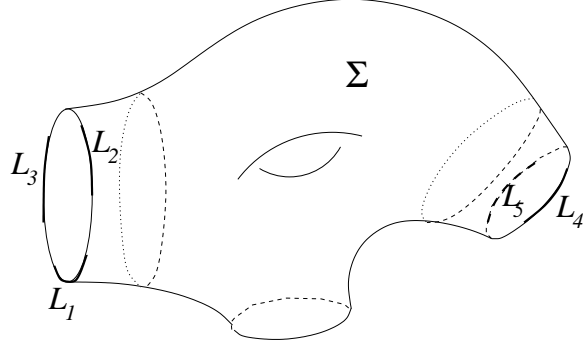


Fig. 13

It is easy to show that the resulting amplitudes are consistent with the gluing of surfaces. For unlabeled surfaces, this is a well known fact. For surfaces glued along two unlabeled boundary intervals in boundary components with labels, there are two different cases. If the glued intervals are in two different boundary components, then the consistency boils down to the case of Fig. 14 and it follows with the use of (6.24).

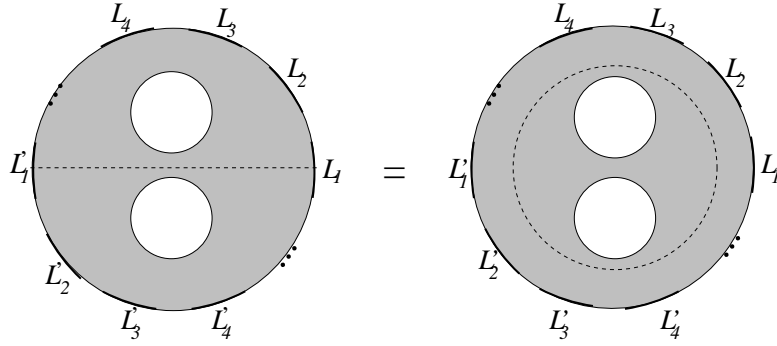


Fig. 14

The case when one glues two intervals in the same boundary components may be similarly reduced to the check that the gluings of Fig. 15 give the same result.

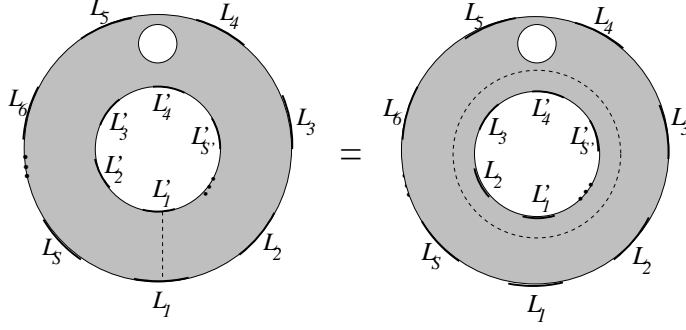


Fig. 15

The first one leads to the amplitude

$$a \otimes a_{L_1 L_2}^1 \otimes \cdots \otimes a_{L_S L_1}^S \otimes b_{L'_1 L'_2}^1 \otimes \cdots \otimes b_{L'_{S'} L'_1}^{S'} \mapsto \sum_A \langle a_{L_1 L_2}^1 \cdots a_{L_S L_1}^S p_{L_1 L'_1}^A b_{L'_1 L'_2}^1 \cdots b_{L'_{S'} L'_1}^{S'}, a p_{L'_1 L_1}^A \rangle_b. \quad (6.26)$$

The second one results in

$$a \otimes a_{L_1 L_2}^1 \otimes \cdots \otimes a_{L_S L_1}^S \otimes b_{L'_1 L'_2}^1 \otimes \cdots \otimes b_{L'_{S'} L'_1}^{S'} \mapsto \sum_{\gamma} \langle a_{L_1 L_2}^1 \cdots a_{L_{S-1} L_S}^{S-1}, p_{\gamma} a_{L_S L_1}^S \rangle_b \langle b_{L'_1 L'_2}^1 \cdots b_{L'_{S'-1} L'_{S'}}^{S'-1}, p'_{\gamma} a_{L'_{S'} L'_1}^{S'} \rangle_b. \quad (6.27)$$

The equality of both expressions follows from (6.17).

Conversely, the amplitudes (6.21) of a two-dimensional boundary topological field theory define the data (6.20). First, the amplitudes of Fig. 6 determine the unity, the bilinear form and the product in \mathcal{H} . The commutativity of the latter follows from the homeomorphism covariance of the amplitudes that allows to permute the two inner discs of the third surface of Fig. 6. The associativity of the product in \mathcal{H} results from the equality of the two ways to glue the amplitudes for the sphere without four discs presented in Fig. 16.

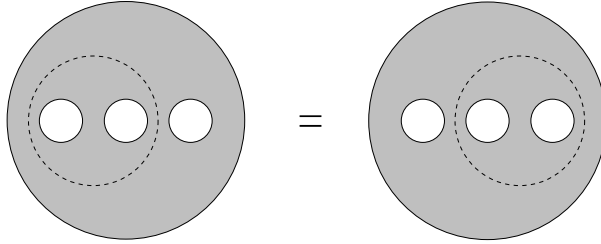


Fig. 16

The first of the relations (6.1) is equivalent to the normalization of the amplitude of the sphere S^2 to 1 and the second follows again from the homeomorphism-covariance of the amplitudes. Similarly, the amplitudes of the first two surfaces of Fig. 10 give the unit elements $1_{LL} \in \mathcal{H}_{LL}$ and the bilinear form pairing $\mathcal{H}_{L_1L_2}$ and $\mathcal{H}_{L_2L_1}$. The amplitude of the third surface applied to $1 \in \mathcal{H}$, together with the bilinear form $\langle \cdot, \cdot \rangle_b$, determine the product $\mathcal{H}_{L_1L_2} \times \mathcal{H}_{L_2L_3} \rightarrow \mathcal{H}_{L_1L_3}$ in such a way that the cyclic invariance (6.14) holds. The associativity is proved similarly as before by equating two ways of gluing a disc with four labeled boundary intervals from pairs of discs with three labeled boundary intervals, see Fig. 17.

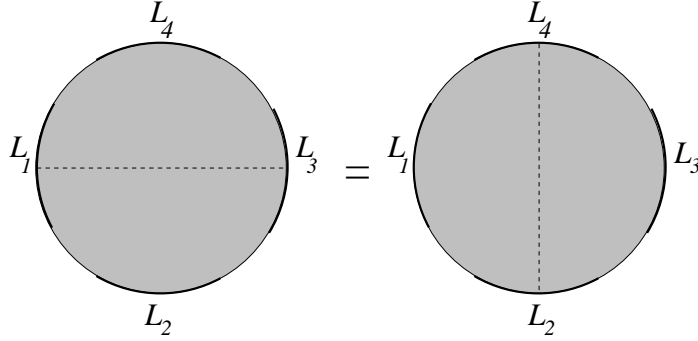


Fig. 17

The action of the elements of the bulk space \mathcal{H} on the boundary space \mathcal{H}_b is obtained from the amplitude of the annulus of Fig. 18

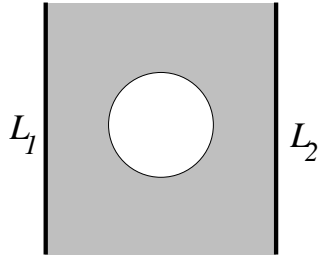


Fig. 18

with the use of the bilinear form $\langle \cdot, \cdot \rangle_b$. By definition, this action preserves the subspaces $\mathcal{H}_{L_1L_2} \subset \mathcal{H}_b$. The proof that it defines a representation of the commutative algebra \mathcal{H} in \mathcal{H}_b follows from Fig. 19.

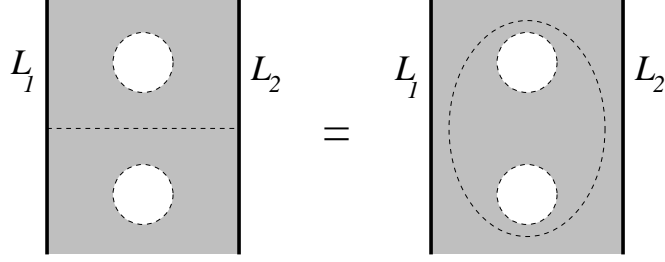


Fig. 19

Similarly, relations (6.10) follow from Fig. 20 and (6.17) from Fig. 15 with $S = S' = 1$.

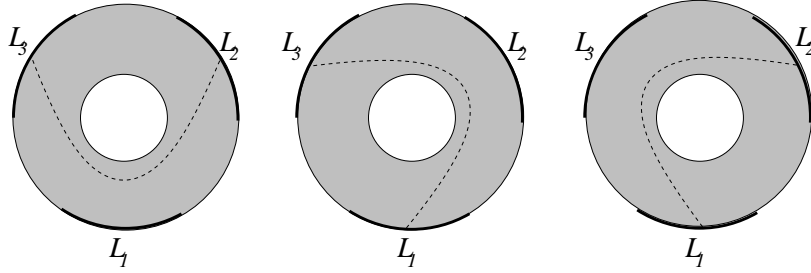


Fig. 20

We obtain this way the algebraic structure (6.20) possessing all the four properties listed.

We shall call a two-dimensional topological field theory unitary if there exist anti-linear involutions $C : \mathcal{H} \rightarrow \mathcal{H}$ and $C_b : \mathcal{H}_b \rightarrow \mathcal{H}_b$ with $C_b(\mathcal{H}_{L_1 L_2}) = \mathcal{H}_{L_2 L_1}$ such that the sesquilinear forms $\langle C \cdot, \cdot \rangle$ and $\langle C_b \cdot, \cdot \rangle_b$ define scalar products on \mathcal{H} and \mathcal{H}_b and that

5. $C(ab) = (Ca)(Cb)$, $C_b(\underline{a}\underline{b}) = (C_b \underline{b})(C_b \underline{a})$, $C_b(a \underline{b}) = (Ca)(C_b \underline{b})$ for $a, b \in \mathcal{H}$ and $\underline{a}, \underline{b} \in \mathcal{H}_b$.

The last three properties guarantee that

$$\mathcal{I}_{-\Sigma} = \overline{\left(\bigotimes'_n C \right) \otimes \left(\bigotimes''_{n,s} C_b \right)}, \quad (6.28)$$

where $-\Sigma$ denotes the surface with the reversed orientation and, conversely, they follow from (6.28). For the G/G theory, one may take for C the complex conjugation of functions of integrable weights and for C_b the hermitian conjugation of linear transformations in $H_{L_0 L_\pi}$, see (6.8), relative to some scalar product in the spaces $\mathcal{H}_{\lambda \eta \zeta}$ of three-point conformal blocks. One obtains then a unitary topological field theory provided the sign in (6.12) is chosen so that $\pm S_0^\zeta > 0$.

Recall that, due to (6.10), the elements $a 1_{LL}$ for $a \in \mathcal{H}$ are in the center of \mathcal{H}_{LL} . Following [23], we shall call the boundary condition L irreducible if all the elements of the center of \mathcal{H}_{LL} are of this form. This is the case in the G/G theory.

To each boundary condition L one may associate a state $a_L \in \mathcal{H}$ using the amplitude of the second surface of Fig. 12 and the bilinear form on \mathcal{H} . Explicitly, a_L is defined by demanding that

$$\langle 1_{LL}, a 1_{LL} \rangle_b = \langle a_L, a \rangle \quad (6.29)$$

for all $a \in \mathcal{H}$. We shall call the family of boundary conditions (L) complete if the states (a_L) span \mathcal{H} . In the G/G theory, for $L = (\lambda, \eta)$,

$$a_L(\hat{\zeta}) = N_{\lambda\bar{\eta}}^{\zeta} (S_0^{\zeta})^{-1} \quad (6.30)$$

and the completeness is easy to see by taking, for example, the conditions with $L = (\lambda, 0)$. On the other hand, the diagonal subfamily of boundary conditions corresponding to $L = (\lambda, \lambda)$ is, in general, not complete since not all integrable weights appear in the fusion of pairs of complex conjugate weights (e.g. for $G = SU(2)$, $a_{(j,j)}(\hat{j}')$ vanishes for half-integer spins j').

The bulk topological theories may be perturbed by “massive” topological perturbations. For example, in the $SU(2)/SU(2)$ model such perturbations permit to establish a relation with twisted minimal $N = 2$ topological theories. One of the interesting open problems for future research is how to extend such relations to the case of the boundary G/G theory.

Appendix A

When expressed in terms of the left and right movers, the symplectic form of the bulk G/H coset theory becomes:

$$\begin{aligned} \Omega^{G/H} = & \frac{k}{4\pi} \int_0^{2\pi} \text{tr} \left[(g_\ell^{-1} \delta g_\ell) \partial_y (g_\ell^{-1} \delta g_\ell) - (g_r^{-1} \delta g_r) \partial_y (g_r^{-1} \delta g_r) \right] dy \\ & - \frac{k}{4\pi} \text{tr} \left[(\delta \rho) \rho^{-1} \left(((\delta g_\ell) g_\ell^{-1})(0) - ((\delta g_r) g_r^{-1})(0) \right) \right. \\ & + \left(g_\ell(0)^{-1} (\delta \rho) \rho^{-1} g_\ell(0) - g_r(0)^{-1} (\delta \rho) \rho^{-1} g_r(0) \right. \\ & \left. \left. - (g_\ell^{-1} \delta g_\ell)(0) + (g_r^{-1} \delta g_r)(0) \right) (\delta \gamma) \gamma^{-1} \right]. \end{aligned} \quad (A.1)$$

The expression for the bulk WZW model symplectic form Ω^{WZW} may be obtained from the latter by setting ρ identically to 1.

Similarly, the expression in terms of the left-mover g_ℓ for the boundary G/H model symplectic form becomes:

$$\Omega_{M_0 M_\pi}^{G/H} = \frac{k}{4\pi} \int_0^{2\pi} \text{tr} \left[(g_\ell^{-1} \delta g_\ell) \partial_y (g_\ell^{-1} \delta g_\ell) \right] dy + \frac{k}{4\pi} \text{tr} \left[(\delta n_0) n_0^{-1} (\delta n_\pi) n_\pi^{-1} \right]$$

$$\begin{aligned}
& - (\delta m_0) m_0^{-1} (\delta m_\pi) m_\pi^{-1} - (\delta \rho) \rho^{-1} ((\delta g_\ell) g_\ell^{-1})(0) \\
& + (g_\ell^{-1} \delta g_\ell)(0) (\delta \gamma) \gamma^{-1} - (\delta \rho) \rho^{-1} g_\ell(0) (\delta \gamma) \gamma^{-1} g_\ell(0)^{-1} \\
& + \omega_{\mu_0}^G(m_0) - \omega_{\nu_0}^H(n_0) - \omega_{\mu_\pi}^G(m_\pi) + \omega_{\nu_\pi}^H(n_\pi) \Big] \quad (\text{A.2})
\end{aligned}$$

and the expression for the boundary WZW model symplectic form $\Omega_{\mu_0 \mu_\pi}^{WZW}$ is obtained by setting ρ , n_0 and n_π identically to 1.

Appendix B

Our proof of the fact that the isomorphisms I , I' , $I_{\mu_0 \mu_\pi}$ and $I'_{M_0 M_\pi}$ between the WZW and G/H phase spaces and the CS ones preserve the symplectic structure is based on a direct calculation of the form $\int_\Sigma \text{tr}(\delta A)^2$ and of its counterpart for the gauge field B , very much in the spirit of a similar calculation [2] for closed surfaces. Consider first the bulk case that is somewhat simpler. It is enough to examine the case of the G/H coset theory which for $H = \{1\}$ reduces to the WZW model. With g_A given by (5.2), we have

$$\text{tr}(\delta A)^2 = d \text{tr} \left[(g_A^{-1} \delta g_A) d(g_A^{-1} \delta g_A) \right]. \quad (\text{B.1})$$

Integrating the last expression over the annulus \mathcal{Z} cut along the interval $[\frac{1}{2}, 1]$, see Fig. 21, and using the Stokes theorem, we infer that

$$\begin{aligned}
\frac{k}{4\pi} \int_{\mathcal{Z}} \text{tr}(\delta A)^2 &= \frac{k}{4\pi} \int_0^{2\pi} \text{tr} \left[(g_A^{-1} \delta g_A)(y) \partial_y (g_A^{-1} \delta g_A)(y) \right. \\
&\quad \left. - (g_A^{-1} \delta g_A)(y + w_0) \partial_y (g_A^{-1} \delta g_A)(y + w_0) \right] dy \\
&\quad - \frac{k}{4\pi} \text{tr} \left[(\delta \gamma) \gamma^{-1} \left((g_A^{-1} \delta g_A)(0) - (g_A^{-1} \delta g_A)(w_0) \right) \right], \quad (\text{B.2})
\end{aligned}$$

where the second line is the contribution from the integrals along the cut. Similar expression holds for B , h_B and ρ replacing A , g_A and γ , respectively. Subtracting both formulae, we obtain an expression for the symplectic form of the double CS theory on \mathcal{Z} which may be shown to coincide with the right hand side of (A.1) by using (5.5) and the second equality of (3.9).

The case of the boundary G/H coset model may be treated similarly. We define for z in the unit disc D cut along the sub-interval $[-\frac{1}{2}, 1]$ of the real axis

$$\tilde{g}_A(z) = P e^{\int_z^{+i0} A}, \quad \tilde{h}_B(z) = P e^{\int_z^{+i0} B}. \quad (\text{B.3})$$

Note that for $y \in (0, 2\pi)$ we have the equalities $\tilde{g}_A(e^{iy}) = g_A(y)$ and $\tilde{h}_B(e^{iy}) = h_B(y)$ for g_A and h_B given by (5.10). Similarly as before,

$$\int_D \text{tr}(\delta A)^2 = \lim_{\epsilon \rightarrow 0} \int_{\partial D_\epsilon} \text{tr} \left[(\tilde{g}_A^{-1} \delta \tilde{g}_A) d(\tilde{g}_A^{-1} \delta \tilde{g}_A) \right], \quad (\text{B.4})$$

where D_ϵ is the cut unit disc without ϵ -discs around $\pm\frac{1}{2}$, see Fig. 21.

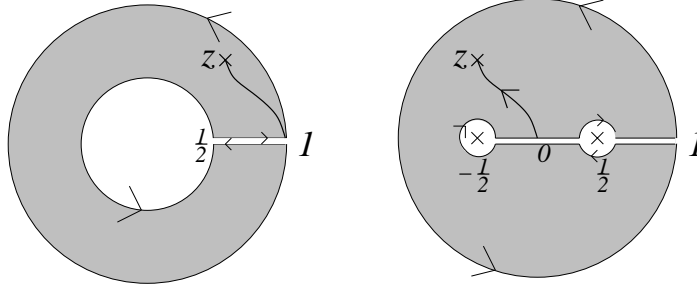


Fig. 21

A tedious but straightforward calculation results in the formula for the symplectic structure of the double CS phase space that coincides with equation (A.2). We leave the details to the reader just stressing that a more direct and conceptual proof of equality between the canonical structures of two-dimensional CFT's and three-dimensional topological field theories would be welcome.

References

- [1] Alekseev, A., Shatashvili, S.: *Quantum groups and WZNW models*, Commun. Math. Phys. **133** (1990), 353-368
- [2] Alekseev, A., Yu., Malkin, A., Z.: *Symplectic structure of the moduli space of flat connections on a Riemann surface*, Commun. Math. Phys. **169** (1995), 99-120
- [3] Alekseev, A. Yu., Schomerus, V.: *D-branes in the WZW model*, Phys. Rev. **D 60** (1999), R061901-R061902
- [4] Atiyah, M. F.: *Topological quantum field theory*, Publ. IHES **68** (1989), 175-186
- [5] K. Bardakçi, E. Rabinovici, B. S aring, *String models with $c < 1$ components*, Nucl. Phys. **B 299** (1988), 151-182
- [6] Cardy, J.: *Boundary conditions, fusion rules and the Verlinde formula*, Nucl. Phys. **B 324** (1989), 581-596
- [7] Cattaneo, A. S., Felder, G.: *A path integral approach to the Kontsevich quantization formula*, Commun. Math. Phys. **212** (2000), 591-611
- [8] Elitzur, S., Moore, G., Schwimmer, A., Seiberg, N., *Remarks on the canonical quantization of the Chern-Simons-Witten theory*, Nucl. Phys. **B 326** (1989), 104-134

- [9] Falceto, F., Gawędzki, K.: *Chern-Simons States at Genus One*, Commun. Math. Phys. **159** (1994), 549-579
- [10] Falceto, F., Gawędzki, K.: *Boundary gauged WZW model and topological Poisson-Lie sigma model*, in preparation
- [11] Felder, G., Fröhlich, J., Fuchs, J., Schweigert, C.: *Conformal boundary conditions and three-dimensional topological field theory*, Phys. Rev. Lett. **84** (2000), 1659-1662
- [12] Felder, G., Fröhlich, J., Fuchs, J., Schweigert, C.: *Correlation functions and boundary conditions in RCFT and three-dimensional topology*, arXiv:hep-th/9912239
- [13] Fuchs, J., Schellekens, B., Schweigert, C.: *The resolution of field identification fixed points in diagonal coset theories*, Nucl.Phys. **B 461** (1996), 371-406
- [14] Gawędzki, K.: *Classical origin of quantum group symmetries in Wess-Zumino-Witten conformal field theory*, Commun. Math. Phys. **139** (1991), 201-213
- [15] Gawędzki, K.: *Conformal field theory: a case study*. In: *Conformal Field Theory*, Frontiers in Physics 102, eds. Nutku, Y., Sağlıoğlu, C., Turgut, T., Perseus Publishing, Cambridge Ma. 2000, pp. 1-55
- [16] Gawędzki, K., Kupiainen, A.: *G/H conformal field theory from gauged WZW model*, Phys. Lett. **B 215** (1988), 119-123
- [17] Gawędzki, K., Kupiainen, A.: *Coset construction from functional integrals*, Nucl. Phys. **B 320** (1989), 625-668
- [18] Gawędzki, K., Todorov, I., Tran-Ngoc-Bich, P.: *Canonical quantization of the boundary Wess-Zumino-Witten model*, arXiv:hep-th/0101170
- [19] Gawędzki, K., Reis, N., Schomerus, V.: in preparation
- [20] Goddard, P., Kent, A., Olive, D.: *Unitary representations of the Virasoro and super-Virasoro algebras*, Commun. Math. Phys. **103** (1986), 105-119
- [21] Hori, K.: *Global aspects of gauged Wess-Zumino-Witten models*, Commun. Math. Phys. **182** (1996), 1-32
- [22] D. Karabali, Q. Park, H. J. Schnitzer, Z. Yang, *A GKO construction based on a path integral formulation of gauged Wess-Zumino-Witten actions*, Phys. Lett. **B 216**, 307-312 (1989)
- [23] Lazaroiu, C. I.: *On the structure of open-closed topological field theory in two-dimensions*, Nucl.Phys. **B 603** (2001), 497-530
- [24] Kirillov, A.: *Elements of the Theory of Representations*, Berlin, Heidelberg, New York, Springer 1975
- [25] Maldacena, J., Moore, G., Seiberg, N.: *Geometrical interpretation of D-branes in gauged WZW models*, arXiv:hep-th/0105038

- [26] Moore, G.: *Some Comments on Branes, G-flux, and K-theory*, Int.J.Mod.Phys. A16 (2001), 936-944
- [27] Moore, G., *Santa Barbara lectures*, <http://online.itp.ucsb.edu/online/mp01>
- [28] Moore, G., Seiberg, N.: *Taming the conformal Zoo*, Phys. Lett. **B 220** (1989), 422-430
- [29] Petkova, V., B., Zuber, J.-B.: *Conformal boundary conditions and what they teach us*, arXiv:hep-th/0103007
- [30] Schellekens, A. N., Yankielowicz, S.: *Field identification fixed points in the coset construction*, Nucl. Phys. **B 334** (1990), 67-102
- [31] Schwarz, A.: *The partition function of degenerate quadratic functional and Ray-Singer invariants*, Lett. Math. Phys. **2** (1978), 247-252
- [32] E. Verlinde, *Fusion rules and modular transformations in 2D conformal field theory*, Nucl. Phys. B **300** [FS **22**] (1988), 360-376
- [33] Witten, E.: *Non-abelian bosonization in two dimensions*, Commun. Math. Phys. **92** (1984), 455-472
- [34] Witten, E.: *Quantum field theory and the Jones polynomial*, Commun. Math. Phys. **121** (1989), 351-399